CODIMENSION TWO SOULS AND CANCELLATION PHENOMENA

IGOR BELEGRADEK, SŁAWOMIR KWASIK, AND REINHARD SCHULTZ

Abstract. For each $m \geq 0$ we find an open $(4m+9)$-dimensional simply-connected manifold admitting complete nonnegatively curved metrics whose souls are non-diffeomorphic, homeomorphic, and have codimension 2. We give a diffeomorphism classification of the pairs (N, soul) when N is a nontrivial complex line bundle over $S^7 \times \mathbb{CP}^2$: up to diffeomorphism there are precisely three such pairs, distinguished by their non-diffeomorphic souls.

1. Introduction

In dimension 7 there are several examples of closed Riemannian manifolds of $\sec \geq 0$ that are homeomorphic and non-diffeomorphic. Historically, the first such example is an exotic 7-sphere discovered by Gromoll-Meyer as the biquotient $Sp(2)/\!/Sp(1)$ [GM74]. Other examples include some homotopy 7-spheres with metrics of $\sec \geq 0$ constructed by Grove-Ziller [GZ00], and examples found among Eschenburg spaces and Witten manifolds by Kreck-Stolz [KS88, KS93] (see also [CEZ07]).

Our main result gives first examples of this kind in dimensions > 7, e.g. we show that $S^7 \times \mathbb{CP}^{2m}$ is not diffeomorphic to $Sp(2)/\!/Sp(1) \times \mathbb{CP}^{2m}$. This is proved via a delicate argument which mixes surgery with homotopy-theoretic considerations of [Sch73, BS74, Sch87].

Recall that for any integer d there is a unique oriented homotopy 7-sphere $\Sigma^7(d)$ that bounds a parallelizable manifold of signature $8d$ [KM63]. Here $\Sigma^7(0) = S^7$, and $\Sigma^7(1) = Sp(2)/\!/Sp(1)$ generates $bP_8 \cong \mathbb{Z}_{28}$, the group of oriented homotopy 7-spheres, which all bound parallelizable manifolds.

Grove-Ziller [GZ00] constructed cohomogeneity 1 metrics of $\sec \geq 0$ on all exotic 7-spheres that are linear S^3-bundles over S^4. A classification of such exotic spheres by Eells-Kuiper [EK62] then implies that $\Sigma^7(d)$ admits a metric of $\sec \geq 0$ if $d \equiv \frac{h(h-1)}{2}$ mod 28 for some integer h. Since $\Sigma^7(-d)$ and $\Sigma^7(d)$ are orientation reversing diffeomorphic, we summarize that unoriented

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diffeomorphism classes of homotopy 7-spheres are represented by $\Sigma^7(d)$'s with $0 \le d \le 14$ where such $\Sigma^7(d)$ is known to admit a metric of $\sec \ge 0$ if $d \notin \{2,5,9,12\}$. We prove

Theorem 1.1. *If m, d, d' are integers and $d-d'$ is odd, then $\Sigma^7(d)\times\mathbb{CP}^{2m}$ is not diffeomorphic to $\Sigma^7(d')\times\mathbb{CP}^{2m}$.*

The proof of Theorem 1.1 occupies most of the paper and is sketched in Section 4. As we see in Lemma 12.2 below, $\Sigma^7(d)\times\mathbb{CP}^{2m}$ and $\Sigma^7(d')\times\mathbb{CP}^{2m}$ are diffeomorphic if either $d-d'$ or $d+d'$ is divisible by 4, and m is not divisible by 3. For $m=1$ this is optimal:

Theorem 1.2. *If d, d' are integers, then $\Sigma^7(d)\times\mathbb{CP}^2$ is diffeomorphic to $\Sigma^7(d')\times\mathbb{CP}^2$ if and only if either $d-d'$ or $d+d'$ is divisible by 4.*

Therefore, if m is not divisible by 3, then each $\Sigma^7(d)\times\mathbb{CP}^{2m}$ admits a metric of $\sec\ge 0$, and the manifolds $\Sigma^7(d)\times\mathbb{CP}^{2m}$ lie in 2 or 3 unoriented diffeomorphism classes; for $m=1$ they lie in 3 unoriented diffeomorphism classes.

We also show that any manifold that is tangentially homotopy equivalent to $\Sigma^7(d)\times\mathbb{CP}^2$ is diffeomorphic to $\Sigma^7(d')\times\mathbb{CP}^2$ for some d' (see Section 13).

Studying whether products of $\mathbb{CP}^n$ with homotopy spheres are diffeomorphic goes back to Browder who showed its relevance to constructing smooth semifree circle actions on homotopy $(2k+7)$-spheres. In particular, by results in [Bro68], Theorem 1.1 applied for $d=1$ immediately implies:

Corollary 1.3. *Given an odd integer d and positive integer k, the exotic sphere $\Sigma^7(d)$ is diffeomorphic to the fixed point set of a smooth semifree circle action on a homotopy $(2k+7)$-sphere if and only if k is even.*

The result is new for $k=3$. The corresponding result for $k\ge 5$ was sketched by Schultz in [Sch87, Theorem III]. The case $k=1$ follows from a result of Hsiang [Hsi64, Theorem II]. If k is even, [Bro68, Theorem 6.1] implies that any $\Sigma^7(d)\times\mathbb{CP}^{k-1}$ is diffeomorphic to $S^7\times\mathbb{CP}^{k-1}$, and that any homotopy 7-sphere can be realized as a fixed point set of a smooth semifree S^1-action on a homotopy $(2k+7)$-sphere.

Any open complete manifold N of $\sec\ge 0$ is diffeomorphic to the total space of a normal bundle to a compact totally geodesic submanifold, called a soul [CG72]. The diffeomorphism class of a soul may depend on the metric, and this dependence has been investigated in [Bel03, KPT05] and most recently in [BKS] where the reader can find further motivation and background. In particular, in [BKS] we systematically searched for open manifolds admitting metrics with non-diffeomorphic souls of lowest possible codimension. To this end we show:

Theorem 1.4. *For each $m \geq 0$ there exists an open $(4m+9)$-dimensional simply-connected manifold N admitting complete metrics of* sec ≥ 0 *whose souls are non-diffeomorphic, homeomorphic, and have codimension 2.*

Codimension 2 is the lowest possible if $\dim(N) \geq 6$ because by the h-cobordism theorem, in a simply-connected manifold open manifold of dim ≥ 6 all codimension 1 souls are diffeomorphic. Letting N in Theorem 1.4 be the product of $\mathbb{R}^2$ with manifolds in Theorem 1.1 cannot work because a closed simply-connected manifold of dimension ≥ 5 can be recovered up to diffeomorphism from its product with $\mathbb{R}^2$. Instead, we find nontrivial $\mathbb{R}^2$-bundles over manifolds in Theorem 1.1 that admit metrics of sec ≥ 0 and have diffeomorphic total spaces. The same reasoning works for $\mathbb{R}^2$-bundles over Eschenburg spaces or Witten manifolds that are homeomorphic and non-diffeomorphic, which covers the case $m = 0$ in Theorem 1.4.

Let $\mathfrak{M}^{k,c}_{\sec\geq 0}(N)$ denote the moduli space of complete metrics of sec ≥ 0 on N with topology of C^k-convergence on compact subsets, where $0 \leq k \leq \infty$. Suppose N admits a complete metric with sec ≥ 0 whose soul has non-trivial normal Euler class. Then it was shown in [KPT05] that metrics with non-diffeomorphic souls lie in different components of $\mathfrak{M}^{k,c}_{\sec\geq 0}(N)$; more generally, the authors showed in [BKS] that associating to the nonnegatively curved metric g the diffeomorphism type of the pair $(N, \text{soul of } g)$ defines a locally constant function on $\mathfrak{M}^{k,c}_{\sec\geq 0}(N)$; thus non-diffeomorphic pairs correspond to metrics in different components of the moduli space. Since the souls in Theorem 1.4 have nontrivial normal Euler class, we get:

Corollary 1.5. *$\mathfrak{M}^{k,c}_{\sec\geq 0}(N)$ is not connected for N as in Theorem* 1.4*.*

Given an open manifold N admitting a complete metric of sec ≥ 0 with soul S_0, an attractive goal is to obtain a diffeomorphism classification of pairs (N, S) where S is a soul of some complete metric of sec ≥ 0 on N. Here we focus on the case when S_0 is a simply-connected, has codimension 2, and dimension ≥ 5. If S_0 has trivial normal bundle, and S is any other soul in N, then the pairs (N, S), (N, S_0) are diffeomorphic (see [BKS]).

To our knowledge the results below are the first instances of diffeomorphism classification of pairs (N, S) in which not all pairs are diffeomorphic.

Theorem 1.6. *The total space N of any nontrivial complex line bundle over $\mathbb{CP}^2 \times S^7$ admits 3 complete nonnegatively curved metrics with pairwise non-diffeomorphic souls S_0, S_1, S_2 such that for any complete nonnegatively curved metric on N with soul S, there exists a self-diffeomorphism of N taking S to some S_i.*

Here S_i is isometric to the product $\mathbb{CP}^2 \times \Sigma^7(3i)$ where the second factor is given a metric of $\sec \geq 0$ by [GZ00].

One can prove similar result for 7-dimensional souls which are certain Witten manifolds. Recall that the *Witten manifold* $M_{k,l}$ is the total space of an oriented circle bundle over $\mathbb{CP}^2 \times \mathbb{CP}^1$ with Euler class given by $(l,k) \in H^2(\mathbb{CP}^2) \oplus H^2(\mathbb{CP}^1)$ where l, k are nonzero coprime integers. In [KS88, Theorem B] Kreck-Stolz classified Witten manifolds $M_{k,l}$ up to oriented homeomorphism and diffeomorphism in terms of k, l, and the above definition of $M_{k,l}$ easily implies that $M_{-k,-l}$ is oriented diffeomorphic to $M_{k,l}$ with the opposite orientation, so one also has a (unoriented) diffeomorphism classification of Witten manifolds. As remarked after [KS88, Theorem C], the homeomorphism type of $M_{k,l}$ consists of Witten manifolds if $l \equiv 0 \bmod 4$ and $l \equiv 0, 3, 4 \bmod 7$. We prove:

Theorem 1.7. *For nonzero coprime integers k, l with $l \equiv 0, 3, 4 \bmod 7$ and $l \equiv 0 \bmod 4$, let N be the total space of a nontrivial $\mathbb{R}^2$-bundle over $M_{k,l}$.*
(1) *If the Witten manifold $M_{k',l'}$ is homeomorphic to $M_{k,l}$, then N has a complete metric of* $\sec \geq 0$ *whose soul $S_{k',l'}$ is diffeomorphic to $M_{k',l'}$.*
(2) *For any complete metric of* $\sec \geq 0$ *on N with soul S the pair (N, S) is diffeomorphic to $(N, S_{k',l'})$ for some $S_{k',l'}$ as in* (1).

The proof of Theorems 1.6 and 1.7 hinges on the following three observations.

- If S, S' are homeomorphic, non-diffeomorphic manifolds that are Eschenburg spaces, or Witten manifolds, or products $\Sigma^7(d) \times \mathbb{CP}^{2m}$, then S' is the connected sum of S with a homotopy sphere.
- If S' is a closed simply-connected manifold of dimension ≥ 5, and if S is diffeomorphic to the connected sum of S' and a homotopy sphere, then there are nontrivial $\mathbb{R}^2$-bundles over S, S' with diffeomorphic total spaces (see Theorem 14.1).
- If S, S' are simply-connected souls of codimension 2 and dimension ≥ 5, then S is diffeomorphic to the connected sum of S' and a homotopy sphere (see [BKS, Theorem 1.8]).

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3. On classifying smooth manifolds via surgery

In this section we describe some results of surgery theory that are used throughout this paper paper. Background references for surgery are Wall's book [Wal99],

especially Chapters 3 and 10, recent book of Ranicki [Ran02], and Browder's book [Bro72] for the simply-connected case.

Let M^n be a compact smooth manifold, with or without boundary, where unless stated otherwise both M and ∂M are assumed connected. We also assume that $n \geq 6$ if $\partial M \neq \emptyset$, and $n \geq 5$ otherwise. A *simple homotopy structure on* M is a pair (N, f) consisting of a compact smooth manifold N and a simple homotopy equivalence of manifolds with boundary (in other words, a homotopy equivalence of pairs). Two such structures (N_1, f_1) and (N_2, f_2) are said to be equivalent if there is a diffeomorphism $h : N_1 \to N_2$ such that $f_2 \circ h \simeq f_1$, where again the homotopy is a homotopy of pairs. The set of all such equivalence classes is a pointed set which is called a *structure set* and is denoted by $\mathbf{S}^s(M)$. Its base point is the identity on M, and this pointed set fits into an exact Sullivan-Wall surgery exact sequence

$$\cdots \to [\Sigma(M/\partial M), F/O] \xrightarrow{\sigma} L^s_{n+1}(\pi_1(M), \pi_1(\partial M)) \xrightarrow{\Delta} \mathbf{S}^s(M) \xrightarrow{\mathfrak{q}} [M, F/O] \xrightarrow{\sigma} \dots$$

which can be continued indefinitely to the left, and one step to the right as $\sigma\colon [M, F/O] \to L^s_n(\pi_1(M), \pi_1(\partial M))$. Here $L^s_m(\pi_1(M), \pi_1(\partial M))$ is an abelian group, called the Wall group, that depends only on the (inclusion induced) homomorphism $\pi_1(\partial M) \to \pi_1(M)$, the value of m modulo 4, and the (orientaton) homomorphism $w\colon \pi_1(M) \to \mathbb{Z}_2$, which we omit from the notation. The map Δ is comes from an action of $L^s_{n+1}(\pi_1(M), \pi_1(\partial M))$ on $\mathbf{S}^s(M)$, namely Δ send an element α of the Wall group to the α-image of the structure represented by $\mathbf{id}(M)$. The map $\mathfrak{q}$ from $\mathbf{S}^s(M)$ to the set of homotopy classes $[M, F/O]$ is called the *normal invariant*. The map σ is called the *surgery obstruction map*.

The exactness in the term $\mathbf{S}^s(M)$ means that two simple homotopy structures have equal normal invariants if and only if they are in the same orbit of Δ. Even though $[M, F/O]$ is an abelian group, $\sigma\colon [M, F/O] \to L^s_n(\pi_1(M), \pi_1(\partial M))$ is not necessarily a homomorphism, and exactness at $[M, F/O]$ means that the image of $\mathfrak{q}$ equals to $\sigma^{-1}(0)$; nevertheless, σ becomes homomorphism in the continuation of the surgery sequence to the left starting from $[\Sigma(M/\partial M), F/O] \to L^s_{n+1}(\pi_1(M), \pi_1(\partial M))$ [Wal99, Proposition 10.7].

Although surgery theory in principle yields a diffeomorphism classification for closed manifolds with a fixed homotopy type, it does so indirectly, and only for a few homotopy types a complete classification is known.

If the inclusion $\partial M \to M$ induces a π_1-isomorphism, the relative Wall groups vanish, and $\mathbf{S}^s(M)$ is bijective to $[M, F/O]$ via $\mathfrak{q}$.

If $M = S^n$, then $\mathbf{S}^s(S^n) = \Theta_n$, the set of oriented diffeomorphism classes of homotopy n-spheres which forms a group under connected sum. The subgroup bP_{n+1} of homotopy n-spheres that bound parallelizable manifolds can be identified with the image of the homomorphism $L^s_{n+1}(1) \to \mathbf{S}^s(S^n)$.

If M is closed and simply-connected, then the action of $L^s_{n+1}(1)$ on $\mathbf{S}^s(M)$ factors through the bP_{n+1}-action via connected sum, and for any two homotopy equivalences $f_1, f_2\colon N \to M$ with equal normal invariants, f_1 is the connected sum of f_2 with an orientation-preserving homeomorphism $\Sigma^n \to S^n$ where Σ^n represents an element of bP_{n+1}; in particular, N_1 is diffeomorphic to $N_2 \# \Sigma^n$.

Kervaire-Milnor [KM63] showed that bP_{n+1} is a finite cyclic group which vanishes if n is even, and has order at most 2 if $n = 4r+1$. On the other hand, the order of P_{4r} grows exponentially with r, and this is the case we focus on in the present paper. Each element of bP_{4r} with $r \geq 2$ is represented by a (unique up to orientation-preserving diffeomorphism) homotopy sphere $\Sigma^{4r-1}(d)$ that bounds a parallelizable manifold of signature $8d$. Here $\Sigma^{4r-1}(1) = S^{4r-1}$, and $\Sigma^{4r-1}(1)$ generates bP_{4r}.

We illustrate how surgery works in a setting that plays an important role in this paper. Removing an open disk from the interior of a parallelizable manifold with boundary $\Sigma^{4r-1}(d)$ yields a parallelizable cobordism W^{4r} between $\Sigma^{4r-1}(d)$ and S^k, and hence defines a normal map

$$F\colon (W^{k+1}, \partial W^{k+1}) \to (S^k \times I, S^k \times \partial I)$$

covered by an isomorphism of trivial tangent bundles. The surgery obstruction is preserved by products with $\mathbb{CP}^{2m}$ [Bro72, Theorem III.5.4], so that $\sigma(F \times \mathbf{id}(\mathbb{CP}^{2m})) = \sigma(F) = d$. Let $f\colon U^{4m+4r}(d) \to D^{4m+4r}$ is a (boundary preserving) degree one map, where $U^{4m+4r}(d)$ is a parallelizable manifold that bounds $\Sigma^{4m+4r}(d)$. Taking boundary connected sums of $F \times \mathbf{id}(\mathbb{CP}^{2m})$ and f along the boundary component $S^{4r-1} \times \mathbb{CP}^{2m}$ defines a normal map with zero surgery obstruction, hence it can be turned into a simple homotopy equivalence via surgery, in other words, we get an s-cobordism between $\Sigma^{4r-1}(d) \times \mathbb{CP}^{2m}$ and $(S^{4r-1} \times \mathbb{CP}^{2m}) \# \Sigma^{4m+4r}(d)$, which are therefore diffeomorphic. In summary the following holds:

Fact 3.1. *Let $M = S^{4r-1} \times \mathbb{CP}^{2m}$ where $r \geq 2$. If $h\colon \Sigma^{4r-1}(d) \to S^{4r-1}$ and $H\colon \Sigma^{4m+4r-1}(d) \to S^{4m+4r-1}$ are orientation-preserving homeomorphisms, then the simple homotopy structures*

$$h \times \mathbf{id}(\mathbb{CP}^{2m})\colon \Sigma^{4r-1}(d) \times \mathbb{CP}^{2m} \to S^{4r-1} \times \mathbb{CP}^{2m}$$
$$H \# \mathbf{id}(\mathbf{M})\colon \Sigma^{4m+4r-1}(d) \# M \to S^{4m+4r-1} \# M = M$$

represent the same element $\Delta(d)$ in the structure set $\mathbf{S}^s(M)$.

Determining the kernel of the bP_{n+1}-action on $\mathbf{S}^s(M)$ is a major step in the diffeomorphism classification of closed manifold homotopy equivalent to M. The *homotopy inertia group $I_h(M)$* of an n-manifold M is the group of all $\Sigma \in \Theta_n$ such that the standard homeomorphism $M\#\Sigma \to M$ is homotopic to a diffeomorphism. The kernel of $\Delta\colon L^s_{n+1}(\pi_1(M)) \to \mathbf{S}^s(M)$ is called the

surgery inertia group and denoted $I_\Delta(M)$. If M is closed and simply-connected, then $I_\Delta(M)$ is the preimage of $I_h(M)\cap bP_{n+1}$ under Δ.

In particular, if M is a closed simply-connected $(4r-1)$-manifold, then $d\in\mathbb{Z}=L^s_{4r}(1)$ acts on $\mathbf{S}^s(M)$ by taking connected sum with $\Sigma^{4r-1}(d)$, and $\Delta(d)$ is trivial in $\mathbf{S}^s(M)$ if and only if $\Sigma(d)\in I_h(M)$.

A key ingredient of our work is the proof given in [Sch87, Theorem 2.1] of the following (unpublished) result of Taylor.

Theorem 3.2. (Taylor) *If M is a closed oriented smooth manifold of dimension $4r-1\geq 7$, then the subgroup $I_h(M)\cap bP_{4r}$ of bP_{4r} has index ≥ 2.*

Taylor's theorem is generally optimal, e.g. if $M=S^3\times\mathbb{C}P^{2m}$ with $m\geq 1$, then the index of $I_h(M)\cap bP_{4r}$ in bP_{4r} is 2. (Indeed, if U^4 is the connected sum of D^4 and a $K3$-surface, then U^4 is a parallelizable manifold and signature $-16=-2\cdot 8$, so arguing as in the above proof of Fact 3.1 we see that the simple homotopy structure $\mathbf{id}(S^3\times\mathbb{CP}^{2m})$ represent both $\Delta(0)$ and $\Delta(-2)$ in $\mathbf{S}^s(S^3\times\mathbb{CP}^{2m})$, which means that $\Sigma^{4m+3}(-2)$ lies in $I_h(S^3\times\mathbb{CP}^{2m})$ as claimed.)

On the other hand, [Bro65, Theorem 2.13] implies that $I_h(M)\cap bP_{4r}$ is trivial if M is a simply connected, stably parallelizable closed manifold of dimension $4r-1\geq 7$. In Section 12 we show that $I_h(M)\cap bP_{4r}$ has index 4 in bP_{4r} if $M=S^7\times\mathbb{CP}^2$.

Even though Taylor's theorem ensures that the standard homeomorphism from $\Sigma^7(1)\#(S^3\times\mathbb{CP}^2)$ to $S^3\times\mathbb{CP}^2$ is not homotopic to a diffeomorphism, these two manifolds are diffeomorphic as proved in [MS99, Corollary 4.2].

This naturally brings us to another source of nontrivial elements in $\mathbf{S}^s(M)$: simple homotopy self-equivalences of M that are not homotopic to diffeomorphisms. For the purposes of diffeomorphism classification, any two homotopy structures $f_1,f_2\colon N\to M$ that differ by a simple homotopy self-equivalences of M should be identified, i.e. we need to take the quotient of $\mathbf{S}^s(M)$ by the action of the group $\mathcal{E}^s(M,\partial M)$ of simple homotopy self-equivalences of $(M,\partial M)$ via composition:

$$[h]\cdot[N,f] \;=\; [N,h\circ f].$$

where (N,f) represents a class in $\mathbf{S}^s(M)$ and $h\in\mathcal{E}^s(M,\partial M)$. With rare exceptions the group $\mathcal{E}^s(M,\partial M)$ is extremely hard to compute, even when M is simply-connected; in this case all homotopy equivalences are simple so in agreement with an earlier notation we write $\mathcal{E}$ instead of $\mathcal{E}^s$.

In comparing elements of $\mathbf{S}^s(M)$ that differ by a homotopy self-equivalence the following *composition formula for normal invariants* is handy (see [Sch71, page 144] or [MTW80, Corollary 2.6]):

$$\mathfrak{q}(g\circ h) \;=\; \mathfrak{q}(g) \;+\; (g^*)^{-1}\,\mathfrak{q}(h),$$

where g represents a class in $\mathbf{S}^s(M)$, and h is a homotopy self-equivalence of M, and the $+$ refers to the group structure in $[M, F/O]$ induced by the Whitney sum in F/O.

Another version of surgery theory concerns relative structure sets $\mathbf{S}^s(M \text{ rel } \partial M)$ of simple homotopy structures (N, f) such that $f|_{\partial N}\colon \partial N \to \partial M$ is a diffeomorphism; two such structures (N_1, f_1) and (N_2, f_2) are said to be equivalent if there is a diffeomorphism $h : N_1 \to N_2$ such that $f_2 \circ h \simeq f_1$, are homotopic through the maps that are diffeomorphisms on the boundary. The corresponding surgery sequence is exact for $n \geq 5$:

$$[\Sigma(M/\partial M), F/O] \;\to\; L^s_{n+1}(\,\pi_1(M)\,) \;\to\; \mathbf{S}^s(M \text{ rel } \partial M) \;\to\; [M/\partial M, F/O]$$

Finally, we summarize results on classifying spaces that are used below. There exists a (homotopy) exact sequence of H-spaces

$$O \;\to\; F \;\to\; F/O \;\to\; BO \;\to\; BF. \tag{3.3}$$

where any three consecutive terms in the sequence form a fibration, and where BF is the classifying space for stable fiber homotopy equivalence classes of spherical fibrations. Applying to the sequence the functor of homotopy classes of maps from a cell complex X yields exact sequence of abelian groups:

$$[X, O] \;\to\; [X, F] \;\to\; [X, F/O] \;\to\; [X, BO] \;\to\; [X, BF],$$

where the groups $[X, F]$ and $[X, BF]$ are finite if X is a finite complex.

4. Sketch of the proof of Theorem 1.1

Let $M = S^k \times \mathbb{CP}^{2m}$. If $k = 4r-1 \geq 7$, then the product formula for the surgery obstruction implies that $\Sigma^k(c) \times \mathbb{CP}^{2m}$ is diffeomorphic to $M \# \Sigma^{4m+k}(c)$ for any integer c, so after taking connected sums of $\Sigma^k(d) \times \mathbb{CP}^{2m}$, $\Sigma^k(d') \times \mathbb{CP}^{2m}$ with $\Sigma^{4m+k}(-d')$ it becomes enough to prove that $M \# \Sigma^{4m+k}(d)$ is not diffeomorphic to M for odd d. By Taylor's Theorem 3.2 the obvious homeomorphism $g\colon\ M \# \Sigma^{4m+7}(d) \to M$ is not homotopic to a diffeomorphism, yet there could exist a diffeomorphism $\phi\colon\ M \to M \# \Sigma^{4m+k}(d)$ with ϕ^{-1} in another homotopy class. By exactness of the smooth surgery sequence, g has trivial normal invariant, and hence so does the homotopy self-equivalence $g \circ \phi$. (Knowing that $g \circ \phi$ is a homeomorphism does not really help because by the topological surgery sequence any homotopy self-equivalence of M with trivial normal invariant is homotopic to a homeomorphism).

We start by decomposing any homotopy self-equivalence of $S^k \times \mathbb{CP}^q$ into the composition of a diffeomorphism, which we may ignore, with two homotopy self-equivalences f, f' coming from adjoints of maps in $\pi_k(E_1(\mathbb{CP}^q))$ and $[\mathbb{CP}^q, E_1(S^k)]$, respectively; this argument works for all odd k.

A priori, vanishing of the normal invariant of $\mathfrak{q}(g\circ\phi)$ need not imply that the normal invariants of $\mathfrak{q}(f)$ and $\mathfrak{q}(f')$ vanish; rather by the composition formula for normal invariants, the sum $\mathfrak{q}(f)+f^{*-1}\mathfrak{q}(f')$ vanishes. Still we show that their normal invariants $\mathfrak{q}(f)$ and $\mathfrak{q}(f')$ restrict nontrivially to skeleta of different dimensions, so they cannot cancel, and therefore, $\mathfrak{q}(f)$ and $\mathfrak{q}(f')$ must both vanish.

Then f' can be extended via the fiberwise cone construction to a homotopy self-equivalence $\hat{f}'$ of $D^{k+1}\times\mathbb{CP}^q$, and then it is easy to see that vanishing of $\mathfrak{q}(f')$ implies vanishing of $\mathfrak{q}(\hat{f}')$, and by Wall's $\pi-\pi$-theorem for manifolds with boundary any simple homotopy structure with trivial normal invariant must be trivial, so restricting back to the boundary shows that f' is homotopic to a diffeomorphism.

To analyze f arising as adjoint of a map in $\pi_k(E_1(\mathbb{CP}^q))$ we need to specialize to $k=7$. For $q\geq 3$, it was shown in [Sch87] that vanishing of $\mathfrak{q}(f)\in[S^7\times\mathbb{CP}^q, F/O]$ implies that f is homotopic to a diffeomorphism. A number of computational details in the proof in [Sch87] are omitted, so for completeness we fill them here with help of [Sch73, BS74] as follows. Results of [Sch73] imply that for $q\geq 3$ the group $\pi_7(E_1(\mathbb{CP}^q))$ stabilizes, and by [BS74] this stable group is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}_2$ where the $\mathbb{Z}$-factor corresponds to the homotopy self-equivalences that come from $U_{q+1}\subset E_1(\mathbb{CP}^q)$, and which are homotopic to diffeomorphisms. On the other hand, following an idea in [Sch87] we show that the homotopy self-equivalence corresponding to the $\mathbb{Z}_2$-factor has nontrivial normal invariant. In summary, the elements in the $\mathbb{Z}$-factor are precisely the ones that give rise to homotopy self-equivalences with trivial normal invariant, which are in fact homotopic to diffeomorphisms.

In dealing with the case $q=2$ a key tool is a spectral sequence in [Sch73] that converges to homotopy groups of $E_1(\mathbb{CP}^q)$. In contrast to the case $q\geq 3$ the group $\pi_7(E_1(\mathbb{CP}^2))$ is unstable, and we use the above spectral sequence to show that suitably defined "stabilization" homomorphism from $\pi_7(E_1(\mathbb{CP}^2))$ to $\pi_7(E_1(\mathbb{CP}^3))$ has image of order 2, and kernel of order ≤ 2. The spectral sequence computation exploits certain low-dimensional phenomena, and involves analyzing various maps between homotopy groups of spheres.

If a homotopy self-equivalence f comes from a stably nontrivial element of $\pi_7(E_1(\mathbb{CP}^2))$, then we show that $\mathfrak{q}(f)$ is nontrivial, so this case does not happen. Thus f comes from a stably trivial element $\pi_7(E_1(\mathbb{CP}^2))$. In general, any homotopy self-equivalence coming from an element of $\pi_7(E(\mathbb{CP}^q))$ is canonically tangential, and we can arrange that it fixes $\{*\}\times\mathbb{CP}^q$ pointwise. This allows to show that the map from $\pi_7(E_1(\mathbb{CP}^q))$ to the structure set of $S^7\times\mathbb{CP}^q$ factors through a homomorphism of $\pi_7(E_1(\mathbb{CP}^q))$ into a relative tangential structure set on $D^7\times\mathbb{CP}^q$, where D^7 should be thought of as the complement of a small open disk centered at $*\in S^7$. The fact that f comes from a stably trivial

element of $\pi_7(E_1(\mathbb{CP}^q))$ translates into vanishing of the normal invariant of f in the relative tangential surgery sequence of $D^7 \times \mathbb{CP}^q$ in which all maps are homomorphism. Looking at this sequence, and combining $L^s_{4m+8}(1) = \mathbb{Z}$ with the fact that f comes from a finite order element of $\pi_7(E_1(\mathbb{CP}^q))$ easily yields that f is trivial in the relative tangential structure set, which gives triviality of the ordinary structure set by naturally extending f to $S^7 \times \mathbb{CP}^q$. Thus f is homotopic to a diffeomorphism.

5. Dichotomy Principle and its applications

Understanding normal invariants of simple homotopy self-equivalences of a closed manifold M is an important step towards classifying manifolds in $\mathbf{S}^s(M)$. In case $M = S^7 \times \mathbb{CP}^{2m}$, we prove the following clean result.

Theorem 5.1. (Dichotomy Principle) *If f is a homotopy self-equivalence of $S^7 \times \mathbb{CP}^{2m}$, then f is homotopic to a diffeomorphism if and only if f has trivial normal invariant.*

In other words, there is a dichotomy: either f is homotopic to a diffeomorphism else f is not even normally cobordant to the identity. We shall prove Theorem 5.1 later in Section 11, and now focus on its applications.

Corollary 5.2. *If $M = S^7 \times \mathbb{CP}^{2m}$, then the number of oriented diffeomorphism types of manifolds $\Sigma^7(d) \times \mathbb{CP}^{2m}$ equals to the index of $I_h(M) \cap bP_{4m+8}$ in bP_{4m+8}. Explicitly, $\Sigma^7(d) \times \mathbb{CP}^{2m}$, $\Sigma^7(d') \times \mathbb{CP}^{2m}$ are oriented-preserving diffeomorphic if and only if $\Sigma^{4m+7}(d - d')$ lies in $I_h(M)$.*

Proof. First suppose that $\Sigma^7(d) \times \mathbb{CP}^{2m}$ and $\Sigma^7(d') \times \mathbb{CP}^{2m}$ are orientation-preserving diffeomorphic. By Fact 3.1 this gives an orientation-preserving diffeomorphism of $\Sigma^{4m+7}(d') \# M$ onto $\Sigma^{4m+7}(d) \# M$, and taking connected sum with $\Sigma^{4m+7}(-d')$, we end up with an orientation-preserving diffeomorphism ϕ of M onto $\Sigma^{4m+7}(d - d') \# M$.

On the other hand, if $H\colon \Sigma^{4m+7}(d - d') \to S^{4m+7}$ is an orientation preserving homeomorphism, then the map $g := H \# \mathbf{id}(M)\colon \Sigma^{4m+7}(d - d') \# M \to M$, representing $\Delta(d - d')$ in the structure set, has trivial normal invariant by exactness of the surgery sequence.

By the composition formula for normal invariants $\mathfrak{q}(g \circ \phi)$ is trivial. Then Theorem 5.1 implies that $g \circ \phi$ is homotopic to a diffeomorphism, and so g is homotopic to a diffeomorphism, i.e. $\Sigma^{4m+7}(d - d')$ lies in the homotopy inertia group $I_h(M)$, as claimed.

Conversely, if $\Sigma^{4m+7}(d - d')$ lies in the homotopy inertia group $I_h(M)$, then $\Sigma^{4m+7}(d - d') \# M$ is diffeomorphic to M, so taking connected sum with

$\Sigma^{4m+7}(d')$, and applying Fact 3.1 gives an orientation-preserving diffeomorphism of $\Sigma^7(d)\times\mathbb{CP}^{2m}$ and $\Sigma^7(d')\times\mathbb{CP}^{2m}$.

Finally, the first assertion of the corollary follows because the preimage of $I_h(M)$ under the map $d\to\Sigma^{4m+7}(d)$ is a subgroup of $\mathbb{Z}$ whose index equals to the index of $I_h(M)\cap bP_{4m+8}$ in bP_{4m+8}, and $d-d'$ is in this subgroup if and only if $\Sigma^7(d)\times\mathbb{CP}^{2m}$ and $\Sigma^7(d')\times\mathbb{CP}^{2m}$ are oriented diffeomorphic. □

Remark 5.3. By composing with the product of $\mathbf{id}(\mathbb{CP}^{2m})$ and an orientation-reversing diffeomorphism $\Sigma^7(d')\to\Sigma^7(-d')$, we immediately conclude that $\Sigma^7(d)\times\mathbb{CP}^{2m}$, $\Sigma^7(d')\times\mathbb{CP}^{2m}$ are oriented-reversing diffeomorphic if and only if $\Sigma^{4m+7}(d+d')$ lies in in $I_h(S^7\times\mathbb{CP}^{2m})$.

Remark 5.4. As mentioned in Section 3 the standard homeomorphism from $\Sigma^7(1)\,\#\,(S^3\times\mathbb{CP}^2)$ to $S^3\times\mathbb{CP}^2$ is not homotopic to a diffeomorphism, yet the domain and codomain are diffeomorphic. The proof of Corollary 5.2 then shows that $S^3\times\mathbb{CP}^2$ has a homotopy self-equivalence with trivial normal invariant that is not homotopic to a diffeomorphism.

Proof of Theorem 1.1. If $\Sigma^7(d)\times\mathbb{CP}^{2m}$ and $\Sigma^7(d')\times\mathbb{CP}^{2m}$ are diffeomorphic, then by Corollary 5.2 and Remark 5.3 at least one of the homotopy spheres $\Sigma^{4m+7}(d-d')$, $\Sigma^{4m+7}(d+d')$ lies in the homotopy inertia group $I_h(S^7\times\mathbb{CP}^q)$, which contradicts Taylor's Theorem 3.2 because $d-d'$ and $d+d'=d-d'+2d'$ are odd. □

6. Factorization of self-equivalences of $S^7\times\mathbb{CP}^q$

This section describes a fairly canonical factorization of any homotopy self-equivalence of $S^7\times\mathbb{CP}^{2m}$, with $m\geq 1$, into the composition of a diffeomorphism and two homotopy self-equivalences each arising from a map of one factor into the space of homotopy self-equivalences of the other factor.

Recall that for an arbitrary compact Hausdorff space T, $\mathcal{E}(T)$ denotes the group of all homotopy classes of homotopy self-equivalences of T, and $E_1(T)$ denote the path-component of the identity in the topological monoid of all self-maps of T (with the compact-open topology). If T is homeomorphic to a finite connected cell complex, then $E_1(T)$ has the homotopy type of a CW complex by a result of Milnor [Mil59].

Proposition 6.1. *(i) Let $f: S^k\times\mathbb{CP}^q\to S^k\times\mathbb{CP}^q$ be a homotopy self-equivalence, where $q\geq 1$ and k is odd. Then there is a diffeomorphism $h: S^k\times\mathbb{CP}^q\to S^k\times\mathbb{CP}^q$ such that f and h induce the same automorphism of $H^*(S^k\times\mathbb{CP}^q;\mathbb{Z})$.*
(ii) Let f be above, and assume that f induces the identity on $H^*(S^k\times\mathbb{CP}^q;\mathbb{Z})$. Let $j(S^k): S^k\to S^k\times\mathbb{CP}^q$ and $j(\mathbb{CP}^q):\mathbb{CP}^q\to S^k\times\mathbb{CP}^q$ be slice inclusions

whose images are subspaces of the form $S^k \times \{y_0\}$ *and* $\{x_0\} \times \mathbb{CP}^q$ *respectively, and let* $p(S^k) : S^k \times \mathbb{CP}^q \to S^k$ *and* $p(\mathbb{CP}^q) : S^k \times \mathbb{CP}^q \to \mathbb{CP}^q$ *denote projections onto the respective factors. Then the composites* $p(S^k) \circ f \circ j(S^k)$ *and* $p(\mathbb{CP}^q) \circ f \circ j(\mathbb{CP}^q)$ *are homotopic to the corresponding identity mappings.*

Proof. (i) The ring $H^*(S^k \times \mathbb{CP}^q; \mathbb{Z})$ is generated by the classes of dimensions 2 and k, which also generate cohomology 2nd and kth cohomology groups, so the induced cohomology automorphism f^* of $H^*(S^k \times \mathbb{CP}^q; \mathbb{Z})$ is completely determined by its behavior on the generators in dimensions 2 and k, and it must be multiplication by ± 1 in each case. If χ is the conjugation involution on $\mathbb{CP}^q$, then $\mathbf{id}(S^k) \times \chi$ is multiplication by $+1$ on the k-dimensional generator and multiplication by -1 on the 2-dimensional generator, while if φ is reflection about a standard $(k-1)$- sphere in S^k then $\varphi \times \mathbf{id}(\mathbb{CP}^q)$ is multiplication by -1 on the k-dimensional generator and multiplication by $+1$ on the 2-dimensional generator. Finally, the composition of these maps is multiplication by -1 on both generators. Thus every automorphism of $H^*(S^k \times \mathbb{CP}^q; \mathbb{Z})$ is in fact induced by a diffeomorphism.

(ii) The composite self-map of S^k induces the identity in cohomology and hence is homotopic to the identity; similarly, the composite self-map of $\mathbb{CP}^q$ also induces the identity in cohomology, and a simple obstruction-theoretic argument shows that this composite must also be homotopic to the identity: indeed, the restrictions to $\mathbb{CP}^1$ are homotopic by degree reasons, and the obstructions to extending this to a homotopy of the original maps lie in the groups $H^{2i}\left(\mathbb{CP}^q, \mathbb{CP}^1; \pi_{2i}(\mathbb{CP}^q)\right)$, which are all trivial. □

The next step in analyzing the homotopy self-equivalences of $S^7 \times \mathbb{CP}^q$ can be done in a fairly general context. For the rest of this section X and Y denote path-connected finite cell complexes with base points x_0 and y_0 respectively. which define slice inclusions $i(X), i(Y) : X, Y \to X \times Y$; projections onto the factors are denoted $p(X)$, $p(Y)$.

Let $\mathcal{E}'(X \times Y)$ be the set all classes $[f] \in \mathcal{E}(X \times Y)$ of homotopy self-equivalences such that $p(X) \circ f \circ j(X) \simeq \mathbf{id}(X)$ and $p(Y) \circ f \circ j(Y) \simeq \mathbf{id}(Y)$. If $X \times Y$ is the product of a complex projective space and an odd-dimensional sphere, $\mathcal{E}'(X \times Y)$ equals to the kernel of the action of $\mathcal{E}(X \times Y)$ on cohomology (with one direction given by Proposition 6.1(ii) and the other one is clear as each cohomology class in this product comes from one of the factors).

In general, $\mathcal{E}'(X \times Y)$ need not be a subgroup, yet regardless of whether or not $\mathcal{E}'(X \times Y)$ is a subgroup of $\mathcal{E}(X \times Y)$, there are two important subsets of $\mathcal{E}'(X \times Y)$ that are subgroups, each arising from a map of one factor into the space of homotopy self-equivalences of the other factor. One of these is the image of a homomorphism $\alpha_X \colon [X, E_1(Y)] \to \mathcal{E}'(X \times Y)$ defined as follows. Given a class

in $[X, E_1(Y)]$, choose a base point preserving representative $g : X \to E_1(Y)$; then g is adjoint to a continuous map $g_\# : X \times Y \to Y$ whose restriction to $\{x_0\} \times Y$ is the identity; furthermore, if g' is homotopic to g then $g'_\#$ is homotopic to $g_\#$. (This uses the fact that the adjoint isomorphism of function spaces is a homeomorphism $\mathfrak{F}(A, (B, C)) \cong \mathfrak{F}(A \times B, C)$ where $\mathfrak{F}$ denotes the continuous function space with the compact open topology and A, B, C are compact Hausdorff spaces). Then $\alpha_X([g])$ is defined to be the homotopy class of the (unique) homotopy self-equivalence G that satisfies $p(Y)\circ G = g_\#$ and $p(X)\circ G = p(X)$. Note that $\alpha_X([g])$ lies in $\mathcal{E}'(X \times Y)$ because the assumption that g is base point preserving implies that

$$p(Y)\circ G\circ j(Y) = g_\# \circ j(Y) = \mathbf{id}(Y) \qquad p(X)\circ G\circ j(X) = p(X)\circ j(X) = \mathbf{id}(X).$$

Basic properties of adjoints imply that α_X is a well-defined homomorphism into $\mathcal{E}(X \times Y)$ whose image lies in $\mathcal{E}'(X \times Y)$. Interchanging the roles of X and Y yields a second homomorphism $\alpha_Y : [Y, E_1(X)] \to \mathcal{E}(X \times Y)$ with image in $\mathcal{E}'(X \times Y)$.

Special cases of the following proposition are in the literature (e.g. in [Lev69, 2.5]).

Proposition 6.2. *Every element in $\mathcal{E}'(X\times Y)$ can be decomposed as the product $\alpha_Y(v)\alpha_X(u)$ for some $u \in [X, E_1(Y)]$ and $v \in [Y, E_1(X)]$.*

Proof. Suppose that f represents an element of $\mathcal{E}'(X \times Y)$. By assumption $p(Y)\circ f|_{\{x_0\}\times Y}$ is homotopic to the identity, so after changing f within its homotopy class we may assume $p(Y)\circ f|_{\{x_0\}\times Y}$ is the identity. Let $U\colon X \to \mathfrak{F}(Y, Y)$ be the adjoint of $p(Y)\circ f$. Since $U(x_0) = \mathbf{id}(Y)$ and X is path-connected, the image of U lie in $E_1(Y)$, which lets us think of U as a map $X \to E_1(Y)$. Hence U defines a homotopy self-equivalence g of $X \times Y$ given by $g(x, y) = (x, U(x)(y))$; note that $p(Y)\circ f = p(Y)\circ g$. Let g' be a homotopy inverse of g. Then $p(Y)\circ f\circ g' = p(Y)\circ g\circ g'$ is homotopic to $p(Y)$, and keeping the homotopy equal to $p(X)\circ f\circ g'$ on the X-coordinate defines a homotopy of $f\circ g'$ to a map adjoint to some $V\colon Y \to \mathfrak{F}(X, X)$, and again path-connectedness of Y and the assumption that $p(Y)\circ f|_{X\times\{y_0\}}$ is homotopic to identity imply the image of V is contained in $E_1(X)$. Since f is homotopic to $f\circ g'\circ g$, the homotopy class of f equals to $\alpha_Y(v)\alpha_X(u)$ where u, v are the homotopy classes of U, V, respectively. □

Remark 6.3. Since $X \times Y$ and $Y \times X$ are canonically homeomorphic, Proposition 6.2 easily implies that the classes in $\mathcal{E}'(X \times Y)$ can be also decomposed as $\alpha_X(u')\alpha_Y(v')$ for some u' and v'.

7. A spectral sequence converging to $\pi_7(E_1(\mathbb{CP}^q))$

By Proposition 6.2 if a homotopy self-equivalence of $S^7 \times \mathbb{CP}^q$ induces a trivial map on cohomology, then its homotopy class can be factored as a product of an element in $\pi_7(E_1(\mathbb{CP}^q))$ and an element in $[\mathbb{CP}^q, S^7]$, and in this section we recall and prove some results on $\pi_7(E_1(\mathbb{CP}^q))$.

The space $E_1(\mathbb{CP}^q)$ does not have nice stabilization properties so instead it is convenient to work with the space $F_{S^1}(\mathbb{C}^{q+1})$ of S^1-equivariant self-maps of S^{2q+1}, the unit sphere in $\mathbb{C}^{q+1}$ with the standard S^1-action that has $\mathbb{CP}^q$ as the quotient. The space $F_{S^1}(\mathbb{C}^{q+1})$ is given the compact-open topology; when it is necessary to take a basepoint, the default choice will be the identity. There is an obvious continuous homomorphism from $F_{S^1}(\mathbb{C}^{q+1})$ to $E_1(\mathbb{CP}^q)$ given by passage to quotients, and the results of James [James 1963] show that this map induces π_k-isomorphisms for $k \geq 2$.

An important advantage of $F_{S^1}(\mathbb{C}^{q+1})$ over $E_1(\mathbb{CP}^q)$ is the existence of the *stabilization homomorphism* $s_{q+1}\colon F_{S^1}(\mathbb{C}^{q+1}) \to F_{S^1}(\mathbb{C}^{q+2})$ induced by the double suspension. We denote the infinite stabilization by F_{S^1}. By [BS74] the homotopy group $\pi_k(F_{S^1})$ is isomorphic to the stable homotopy group $\pi_k^{\mathbf{S}}(S\mathbb{CP}^\infty_+)$, where $S\mathbb{CP}^\infty_+$ is the suspension of the disjoint union of $\mathbb{CP}^\infty$ and a point, and moreover by [BS74, Theorem 11.1] the obvious map $U_{q+1} \to F_{S^1}(\mathbb{C}^{q+1})$ is π_k-isomorphism on a direct summand, which has finite index in $\pi_k(F_{S^1}(\mathbb{C}^{q+1}))$. In fact since $S\mathbb{CP}^\infty_+$ is homotopy equivalent to $SCP^\infty \vee S^1$, the group $\pi_7^{\mathbf{S}}(S\mathbb{CP}^\infty_+)$ contains $\pi_7^{\mathbf{S}}(S^1) = \mathbb{Z}_2$ as a direct summand, which implies as we shall see in the proof of Lemma 7.2(4) that $\pi_7(F_{S^1}) = \pi_7(U) \oplus \pi_7^{\mathbf{S}}(S^1) = \mathbb{Z} \oplus \mathbb{Z}_2$. Also as we note below if $q \geq 3$, then s_{q+1} induces a π_7-isomorphism so $\pi_7(F_{S^1}(\mathbb{C}^{q+1})) \cong \pi_7(F_{S^1})$.

On the other hand the group $\pi_7(F_{S^1}(\mathbb{C}^3))$ is unstable, and a goal of this section is to prove the following.

Proposition 7.1. *The map $s_{3*}\colon \pi_7(F_{S^1}(\mathbb{C}^3)) \to \pi_7(F_{S^1}(\mathbb{C}^4)) \cong \pi_7(F_{S^1})$ has kernel of order at most 2, and has image isomorphic to $\mathbb{Z}_2$. If the kernel is nontrivial, then $\pi_7(F_{S^1}(\mathbb{C}^3)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

While Proposition 7.1 does not compute $\pi_7(F_{S^1}(\mathbb{C}^3))$, its conclusion suffices for the purposes of this paper.

General tools for studying homotopy groups of $F_{S^1}(\mathbb{C}^{q+1})$ are spectral sequences developed in [Sch73] and [BS74]. To avoid additional digressions, we only use the spectral sequences described in [Sch73, Sections 1 and 5] and the relations among them. These spectral sequences arise from the long exact homotopy sequences associated to standard filtrations of function spaces and certain classical Lie groups. In the case of $F_{S^1}(\mathbb{C}^{q+1})$, the filtration is given by

the submonoids $\mathrm{Filt}^{(2p-1)} = \mathrm{Filt}^{(2p)}$ of functions that restrict to inclusions on the standard subspheres $S^{2q-2p+1} \subset S^{2q+1}$, where $0 \le p \le q$; by convention, $\mathrm{Filt}^{(2q+1)}$ is the entire space. For the group U_{q+1}, a similar filtration is given by the standardly embedded unitary groups U_p, where p runs through the same set of values. There is an obvious inclusion of U_{q+1} in $F_{S^1}(\mathbb{C}^{q+1})$ which is compatible with these filtrations, and the results of [Sch73, Section 5] relate the spectral sequences for the homotopy groups of these spaces. Other results in [Sch73] describe spectral sequence mappings corresponding to the stabilization maps $F_{S^1}(\mathbb{C}^q) \subset F_{S^1}(\mathbb{C}^{q+1})$. In all cases, the terms $E^1_{s,t}$ are the relative homotopy groups $\pi_{s+t}(\mathrm{Filt}^{(s)}, \mathrm{Filt}^{(s-1)})$, which turn out to be canonically isomorphic to certain homotopy groups of spheres.

Notations. *The spectral sequences for the homotopy groups of the unitary group U_{q+1} which appear in* [Sch73, Theorem 5.2] *will be denoted by $E^r_{s,t}(U_{q+1})$, and the previously discussed spectral sequences for the homotopy groups of the spaces $F_{S^1}(\mathbb{C}^{q+1})$, which are called $G\mathbb{C}^{q+1}$ in* [Sch73]*, will be denoted by $E^r_{s,t}(G\mathbb{C}^{q+1})$; to simplify notations we sometimes are using the notation $G\mathbb{C}^{q+1}$ for the equivariant function space instead of $F_{S^1}(\mathbb{C}^{q+1})$.*

By [Sch73] the spectral sequence $\{E^r_{s,t}(G\mathbb{C}^{q+1})\}$ converges to $\pi_{p+q}(G\mathbb{C}^{q+1})$, and $E^2_{s,t}(G\mathbb{C}^{q+1}) = H_{s-1}(\mathbb{CP}^q, \pi_{t+2q+1}(S^{2q+1}))$, while $\{E^r_{s,t}(U_{q+1})\}$ converges to $\pi_{p+q}(U_{q+1})$, and $E^2_{s,t}(U_{q+1}) = H_{s-1}(\mathbb{CP}^q, \pi_{s+t}(S^s))$. By [Sch73, Theorem 5.2] there is a canonical mapping between the spectral sequences which converges to a map of homotopy groups induced by the inclusion $U_{q+1} \to G\mathbb{C}^{q+1}$, and which on E^2-level correspond to the coefficient homomorphism induced by the $(2q+1-s)$-fold suspension. Similarly, the stabilization homomorphism induces a map between spectral sequences for $G\mathbb{C}^{q+1}$ and $G\mathbb{C}^{q+2}$ which on E^2-level corresponds to the coefficient homomorphism induced by the double suspension and by the inclusion $\mathbb{CP}^q \to \mathbb{CP}^{q+1}$. Now a straightforward computation implies that $\pi_7(F_{S^1}(\mathbb{C}^{q+1}))$ is stable for $q \ge 3$, because then the stabilization homomorphism $E^2_{s,7-s}(G\mathbb{C}^{q+1}) \to E^2_{s,7-s}(G\mathbb{C}^{q+2})$ is an isomorphism.

To prove Proposition 7.1 we need some formulas for differentials in above spectral sequences, where notation for elements in the homotopy groups of spheres is the same as in Toda's book [Tod62].

Lemma 7.2. *In the preceding spectral sequences, one has the following differentials:*
(1) *The differential $d^2_{5,0}(U_3) : \pi_5(S^5) = \mathbb{Z} \to \pi_4(S^3) = \mathbb{Z}_2$ sends the generator of the domain to the generator of the codomain (which is the Hopf map $\eta_4 : S^4 \to S^3$).*
(2) *The differential $d^2_{5,0}(G\mathbb{C}^3) : \pi_5(S^5) = \mathbb{Z} \to \pi_6(S^5) = \mathbb{Z}_2$ sends the generator of the domain to the generator of the codomain (which is the Hopf map*

$\eta_6 : S^6 \to S^5$).
(3) *The differential* $d^2_{5,2}(G\mathbb{C}^3) : \pi_7(S^5) = \mathbb{Z}_2 \to \pi_8(S^5) = \mathbb{Z}_{24}$ *is injective, in fact, it sends the generator of the domain to the unique element in of the codomain of order* 2.
(4) *For each* $q \geq 2$, *the generator of* $E^2_{1,6}(G\mathbb{C}^{q+1}) = \mathbb{Z}_2$ *defines a cycle in* $E^\infty_{1,6}(G\mathbb{C}^{q+1})$, *and the corresponding class in* $\pi_7\left(F_{S^1}(\mathbb{C}^{q+1})\right)$ *is mapped to the unique element of order* 2 *in* $\pi_7(F_{S^1}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

Proof. The validity of (1) follows because this is the only choice of differential which is compatible with the fact that $\pi_4(U_3) = 0$, and (2) then follows because the map from $\pi_4(S^3) = E^2_{3,1}(U_3)$ to $\pi_6(S^5) = E^2_{3,1}(G\mathbb{C}^3)$ is given by double suspension [Sch73, Theorem 5.2, p.70], and this map is bijective [Tod62, Proposition 5.1, p.39].

To establish (3) it is enough to show that $d^2_{5,2}(G\mathbb{C}^3)$ is nontrivial on $\pi_7(S^5) = E^2_{5,2}(G\mathbb{C}^3)$, which is generated by the square η^2 of the Hopf map [Tod62, Proposition 5.3, p. 40]. To this end it helps to use composition operations of the spectral sequence as described in [Sch73, Proposition 1.4, p. 54]. Denoting the identity element of S^5 by 1, and thinking of η^2 as $1\circ\eta^2$, we write $d^2_{5,2}(G\mathbb{C}^3)(\eta^2)$ as $d^2_{5,0}(G\mathbb{C}^3)(1\circ\eta^2)$ which stably equals to $d^2_{5,0}(G\mathbb{C}^3)(1)\circ\eta^2$ because the operation of precompositing with η^2 stably commutes with differentials. Now (2) implies that $d^2_{5,2}(G\mathbb{C}^3)(\eta^2) = \eta\circ\eta^2 = \eta^3$ which has order 2 in $\pi_8(S^5) = E^2_{3,3}(G\mathbb{C}^3)$ [Tod62, formula (5.5), p. 42]; thus $d^2_{5,2}(G\mathbb{C}^3)$ is nontrivial.

It remains to verify (4). First, suppose that $q \geq 3$; then $\pi_{4+2q+1}(S^{2q+1}) = 0$, which implies $E^2_{3,4} = 0$. Using (3) and the stabilization maps for spectral sequences from [Sch73, Theorem 3.2, p. 64] we see that the differentials $d^2_{5,2}(G\mathbb{C}^{q+1}) : \pi_{2q+3}(S^{2q+1}) = \mathbb{Z}_2 \to \pi_{2q+4}(S^{2q+1}) = \mathbb{Z}_{24}$ are nontrivial, so that $E^\infty_{5,2} = 0$. Thus $E^\infty_{s,7-s}(G\mathbb{C}^{q+1})$ is trivial except possibly when $s = 1$ or $s = 7$.

Let us show that $E^\infty_{7,0}(G\mathbb{C}^{q+1}) \cong \mathbb{Z}$. The group $E^2_{s,t}(U_{q+1})$ is zero for even s, and is equal to $\pi_{s+t}(S^s)$ for odd s. Since $\pi_{s+t}(S^s)$ is finite for $t > 0$, the group $E^\infty_{s,t}(U_{q+1})$ is finite for $t > 0$. But $\pi_7(U_{q+1}) = \mathbb{Z}$ for $q \geq 3$, and $\mathbb{Z}$ has only trivial filtrations, so $E^\infty_{s,7-s}(U_{q+1}) = 0$ for $s < 7$ and hence $\pi_7(U_{q+1}) = E^\infty_{7,0}(U_{q+1}) \cong \mathbb{Z}$. As we mentioned above by [BS74] the image of $\pi_7(U_{q+1})$ in $\pi_7(F_{S^1}(\mathbb{C}^{q+1})) \cong \pi_7(F_{S^1})$ is infinite cyclic, hence $E^\infty_{7,0}(G\mathbb{C}^{q+1})$ must be infinite, and since $E^2_{7,0}(G\mathbb{C}^{q+1}) \cong \mathbb{Z}$, we conclude $E^\infty_{7,0}(G\mathbb{C}^{q+1}) \cong \mathbb{Z}$.

Recall that $E^2_{1,6}(G\mathbb{C}^{q+1}) = \pi_{6+2q+1}(S^{2q+1}) \cong \mathbb{Z}_2$, hence if the generator of $E^2_{1,6}(G\mathbb{C}^{q+1})$ were not a cycle in $E^\infty_{1,6}(G\mathbb{C}^{q+1})$, then $E^\infty_{1,6}(G\mathbb{C}^{q+1}) = 0$, which would mean that $\pi_7(F_{S^1}(\mathbb{C}^{q+1})) = E^\infty_{7,0}(G\mathbb{C}^{q+1}) \cong \mathbb{Z}$ which contradicts the fact that $\pi_7(F_{S^1}(\mathbb{C}^{q+1}))$ contains $\pi^{\mathbf{S}}_7(S^1) = \mathbb{Z}_2$. Thus $E^\infty_{1,6}(G\mathbb{C}^{q+1}) \cong \mathbb{Z}_2$ which

gives an order 2 element in $\pi_7(F_{S^1}(\mathbb{C}^{q+1})) \cong \mathbb{Z} \oplus \mathbb{Z}_2$, which is unique and generates the torsion subgroup. This completes the argument for $q \geq 3$.

To recover the case $q = 2$, we note that $E^2_{1,6}(G\mathbb{C}^3) = \pi_{11}(S^5) \cong \mathbb{Z}_2$ and the stabilization map from $\pi_{11}(S^5) \to \pi^{\mathbf{S}}_6 = E^2_{1,6}(G\mathbb{C}^4) = E^\infty_{1,6}(G\mathbb{C}^4)$ is bijective (see [Tod62, Proposition 5.11, p. 46]); thus the nonzero element $E^2_{1,6}(G\mathbb{C}^3)$ survives in $E^\infty_{1,6}(G\mathbb{C}^4)$, which implies that it must survive in $E^\infty_{1,6}(G\mathbb{C}^3)$, as claimed. □

Proof of Proposition 7.1. Lemma 7.2(3) implies that $E^\infty_{s,7-s}(G\mathbb{C}^3) = 0$ except possibly when $s \neq 1, 3$, and from Lemma 7.2(4) we see that $E^\infty_{1,6}(G\mathbb{C}^3) = \mathbb{Z}_2$, which maps injectively into $\pi_7(F_{S^1})$. Therefore, the only uncertainty involves the group $E^\infty_{3,4}(G\mathbb{C}^3)$. Recall that $E^2_{3,4}(G\mathbb{C}^3) = \pi_9(S^5) \cong \mathbb{Z}_2$ [Tod62, Proposition 5.8, p. 43] and $E^2_{3,4}(G\mathbb{C}^4) = 0$. Hence the quotient of $\pi_7(F_{S^1}(\mathbb{C}^3))$ by the subgroup $E^\infty_{1,6}(G\mathbb{C}^3) \cong \mathbb{Z}_2$ is $E^\infty_{3,4}(G\mathbb{C}^3)$, which is a group of order at most 2. Since $E^\infty_{3,4}(G\mathbb{C}^4) = 0$, the stabilization homomorphism maps $\pi_7(F_{S^1}(\mathbb{C}^3))$ onto a (unique) order two subgroup of $\pi_7(F_{S^1}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ that is the image of $E^\infty_{1,6}(G\mathbb{C}^3)$. If $E^\infty_{3,4}(G\mathbb{C}^3) \cong \mathbb{Z}_2$, then the group $\pi_7(F_{S^1}(\mathbb{C}^3))$ has order 4, and it cannot be isomorphic to $\mathbb{Z}_4$ because it has an order 2 element that does not lie in the kernel of the homomorphism into $\pi_7(F_{S^1})$; thus in this case $\pi_7(F_{S^1}(\mathbb{C}^3)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. □

8. Tangential structure sets

The machinery of this section is useful in the study of normal invariants of homotopy self-equivalences of $S^k \times \mathbb{CP}^q$ that come from classes in $\pi_k(E_1(\mathbb{CP}^q)) \cong \pi_k(F_{S^1}\left(\mathbb{C}^{q+1}\right))$, where $k \geq 2$. Below we describe a well-known surgery exact sequence for tangential homotopy equivalences, which in topological category can be found in [MTW80, Section 2].

For $n \geq 5$, a *tangential simple homotopy structure* on an n-manifold X, with or without boundary, is a triple $(N, f, \hat{f})$ such that (N, f) is a simple homotopy structure on X and $\hat{f}\colon \tau_N \to \tau_X$ is a isomorphism of stable tangent bundles that covers f. Two such structures $(N_1, f_1, \hat{f}_1)$, $(N_2, f_2, \hat{f}_2)$ are said to be equivalent if (N_1, f_1), (N_2, f_2) are equivalent as simple homotopy structures, i.e. f_1 and $f_2 \circ h$ are homotopic through maps of pairs $(N_1, \partial N_1) \to (X, \partial X)$ for some diffeomorphism h. (This definition is slightly different in topological category where one has to insists that the differential of the homeomorphism h preserves stable tangent bundles; this holds automatically in smooth category).

We denote the set of equivalence classes of tangential simple homotopy structures on X by $\mathbf{S}^{s,t}(X)$. One then has the tangential surgery exact sequence

which is mapped into the ordinary surgery exact sequence by forgetting the bundle data, forming the commutative diagram below

$$\begin{tikzcd}
 & & & {[X,O]} \arrow[dl] \arrow[d] \\
{[\Sigma(X/\partial X),F]} \arrow[r] \arrow[d] & L^s_{n+1}(\pi_1(X),\pi_1(\partial X)) \arrow[r] \arrow[d, equal] & \mathbf{S}^{s,t}(X) \arrow[r, "\mathfrak{q}^t"'] \arrow[d] & {[X,F]} \arrow[d] \\
{[\Sigma(X/\partial X),F/O]} \arrow[r] & L^s_{n+1}(\pi_1(X),\pi_1(\partial X)) \arrow[r] & \mathbf{S}^{s}(X) \arrow[r] \arrow[dr] & {[X,F/O]} \arrow[d] \\
 & & & {[X,BO]}
\end{tikzcd}$$

where the map $\mathfrak{q}^t$ is called the *refined normal invariant.* The rightmost column corresponds to the fibration sequence $F \to F/O \to BO$. Up to a canonical choice of sign, the map $\mathbf{S}^s(X) \to [X, BO]$ in this exact sequence takes the class represented by (N, f) to the difference $\tau_X - f^{-1*}\tau_N$ of stable vector bundles. The map $[X, O] \to \mathbf{S}^{s,t}(X)$ whose value is given by $(X, 1_X, \Phi)$, where Φ is a stable vector bundle automorphism of $X \times \mathbb{R}^k$ associated to a class in $[X, O_k]$ for $k \gg n$. This results in an exact sequence

$$[X,O] \longrightarrow \mathbf{S}^{s,t}(X) \longrightarrow \mathbf{S}^s(X) \longrightarrow [M,BO].$$

Similarly, one has the tangential relative structure sets $\mathbf{S}^{s,t}(X \text{ rel } \partial X)$ of equivalence classes of the tangential simple homotopy structures that restrict to diffeomorphisms on the boundary, and again in defining equivalent structures one requires that the homotopy is through maps that are diffeomorphisms on the boundary; $\mathbf{S}^{s,t}(X \text{ rel } \partial X)$ fits into the commutative diagram with exact rows:

$$\begin{tikzcd}
L^s_{n+1}(\pi_1(X)) \arrow[r] \arrow[d, equal] & \mathbf{S}^{s,t}(X \text{ rel } \partial X) \arrow[r, "\mathfrak{q}^t"] \arrow[d] & {[X,F]} \arrow[r] \arrow[d] & L^s_n(\pi_1(X)) \arrow[d, equal] \\
L^s_{n+1}(\pi_1(X)) \arrow[r] & \mathbf{S}^{s}(X \text{ rel } \partial X) \arrow[r] & {[X,F/O]} \arrow[r] & L^s_n(\pi_1(X))
\end{tikzcd}$$

For a closed manifold Y and $k \geq 1$, we let $X := D^k \times Y$ and denote

$$\mathbf{S}^s_k(Y) := \mathbf{S}^s(X \text{ rel } \partial X) \quad \text{and} \quad \mathbf{S}^{s,t}_k(X) := \mathbf{S}^{s,t}(X \text{ rel } \partial X).$$

These structure sets have group structures given by the boundary connected sum; moreover, the group structures are abelian if $k \geq 2$. Furthermore, "spacification" techniques (see e.g. [Wei94, Section 3.1]) imply that for $X = D^k \times Y$ all the maps in the above diagram are group homomorphisms; in fact, one can

show that the above surgery sequence comes from the exact homotopy sequence associated to some fibration.

There is a canonical map

$$\Gamma\colon\ \mathbf{S}^s_k(Y) \to \mathbf{S}^s(S^k \times Y)$$

that is obtained by doubling along the boundary. More precisely, given a representative $h : (W, \partial W) \longrightarrow (D^k \times Y, \partial D^k \times Y)$ of a relative structure, where the boundary map ∂h is a diffeomorphism, we take $V = W \cup_{\partial h} D^k \times Y$, where the D^k-factor is given the opposite orientation, and let $V \to S^k \times Y$ be the well-defined map which is given by h on W and the identity on $D^k \times Y$. This construction preserves homotopies through maps that are diffeomorphisms on the boundary, and hence it defines a well-defined map Γ. Also the construction preserves the identity, so Γ preserves the base points (recall that while $\mathbf{S}^s_k(Y)$ is a group, $\mathbf{S}^s(S^k \times Y)$ is merely a pointed set). Note that Γ maps $\mathbf{S}^{s,t}_k(Y)$ to $\mathbf{S}^{s,t}(S^k \times Y)$.

Finally, we relate this abstract machinery to the objectives of this paper. Every element of $\pi_k(E_1(Y))$ can be represented by a map $(D^k, \partial D^k) \to E_1(Y)$ such that a small neighborhood of ∂D^k is mapped to $\mathbf{id}(Y)$. For $k \geq 1$, this yields a group homomorphism $\Psi\colon\ \pi_k(E_1(Y)) \to \mathbf{S}^s_k(Y)$ such that the obvious map $\pi_k(E_1(Y)) \to \mathbf{S}^s(S^k \times Y)$ defined via adjoint can be factored as $\Gamma \circ \Psi$. Moreover, if the image of $\pi_k(E_1(Y)) \to \mathbf{S}^s(S^k \times Y)$ lies in $\mathbf{S}^{s,t}(S^k \times Y)$, then the image of Ψ lies in $\mathbf{S}^{s,t}_k(Y)$.

9. Stably trivial self-equivalences from $\pi_7(E_1(\mathbb{CP}^q))$ are trivial

Suppose $k \geq 2$, fix $\alpha \in \pi_k(F_{S^1}\left(\mathbb{C}^{q+1}\right))$, and let $\widetilde{f}$ be an equivariant self-map of $S^k \times S^{2q+1}$ representing α. Denote by f the corresponding homotopy self-equivalence of the orbit space $S^k \times \mathbb{CP}^q$, Then $\widetilde{f} \times_{S^1} \mathbf{id}(\mathbb{C})$ is a vector bundle self-isomorphism of the canonical line bundle on $S^k \times \mathbb{CP}^q$ which covers f, and since the stable tangent bundle of $\mathbb{CP}^q$ is stably a direct sum of $(q+1)$ copies of the canonical line bundle it follows that $\tilde{f}$ defines an explicit tangential homotopy structure on $S^k \times \mathbb{CP}^q$ which refines the ordinary homotopy structure $(S^k \times \mathbb{CP}^q, f)$. As in the last paragraph of Section 8, this defines a homomorphism $\pi_k\left(F_{S^1}(\mathbb{C}^{q+1})\right) \longrightarrow \mathbf{S}^{s,t}_k(\mathbb{CP}^q)$. In these notations we have:

Proposition 9.1. *Suppose that $k \geq 2$ and $k + 2q \equiv 3 \bmod 4$. If α has finite order in $\pi_k\left(F_{S^1}(\mathbb{C}^{q+1})\right)$ and if the refined normal invariant of f is trivial, then the homotopy self-equivalence f of $S^k \times \mathbb{CP}^q$ is homotopic to a diffeomorphism.*

Proof. The image of α in $\mathbf{S}^{s,t}_k(\mathbb{C}P^q)$ has finite order because the map is a homomorphism. On the other hand, consider the following portion of the tangential

surgery sequence where $X = D^k \times \mathbb{CP}^q$:

$$[\Sigma(X/\partial X), F] \ \rightarrow \ L_{k+2q+1}(\{1\}) \cong \mathbb{Z} \ \rightarrow \ \mathbf{S}^{s,t}_k(\mathbb{C}P^q) \ \rightarrow [X/\partial X, F]$$

The groups $[\Sigma(X/\partial X), F]$, $[X, F]$ are finite, so the map from $\mathbb{Z}$ to $\mathbf{S}^{s,t}_k(\mathbb{C}P^q)$ must be a monomorphism. By exactness α is trivial in $\mathbf{S}^{s,t}_k(\mathbb{C}P^q)$, and hence in $\mathbf{S}^{s}_k(\mathbb{CP}^q)$. Since $\Gamma\colon\ \mathbf{S}^{s}_k(\mathbb{CP}^q) \to \mathbf{S}^s(S^k \times \mathbb{CP}^q)$ preserves the base points, the image of α in $\mathbf{S}^s(S^k \times \mathbb{CP}^q)$ is also trivial. □

Corollary 9.2. *Suppose that $k \geq 2$ and $k \equiv 3 \bmod 4$. Let $\alpha \in \pi_k\left(F_{S^1}(\mathbb{C}^3)\right)$ be a nontrivial element that stabilizes to zero in $\pi_k(F_{S^1})$. Then the image of α in $\mathbf{S}^s(S^k \times \mathbb{CP}^2)$ is trivial.*

Proof. If $X = D^k \times \mathbb{CP}^q$, then $X/\partial X$ is homotopy equivalent to $S^k(\mathbb{CP}^q) \vee S^k$, hence the refined normal invariant takes values in $[S^k(\mathbb{CP}^q) \vee S^k, F]$. We have the following commutative diagram, in which the vertical arrow on the left is induced by the stabilization map from $F_{S^1}(\mathbb{C}^3)$ to $F_{S^1}(\mathbb{C}^{q+1})$ for $q \geq 3$ and the vertical arrow on the right is induced by the inclusion map from $\mathbb{CP}^2$ to $\mathbb{CP}^q$:

(9.3)
$$\begin{tikzcd}
\pi_k(\, F_{S^1}(\mathbb{C}^3)\,) \arrow[r] \arrow[d] & {[S^k(\mathbb{CP}^q) \vee S^k,\ F]} \\
\pi_k(\, F_{S^1}(\mathbb{C}^{q+1})\,) \arrow[r] & {[S^k(\mathbb{CP}^q) \vee S^k,\ F]} \arrow[u]
\end{tikzcd}$$

where horizontal arrows are refined normal invariant precomposed with the homomorphism $\pi_k\left(F_{S^1}(\mathbb{C}^{q+1})\right) \ \longrightarrow \ \mathbf{S}^{s,t}_k(\mathbb{CP}^q)$. Since the image of α under stabilization is trivial, the top horizontal arrow maps α to zero, so that α satisfies the assumptions of the preceding proposition. □

10. Normal invariant of the stable element of order 2

As mentioned in Section 7, the group $\pi_7(F_{S^1})$ has a unique order two element, and in this section we show that the corresponding homotopy self-equivalence f of $S^7 \times \mathbb{CP}^q$, $q \geq 3$ has nontrivial normal invariant. A proof of this was briefly sketched in [Sch87]; here we provide details.

It is shown in Section 9 that f is tangential. Let γ denote the normal bundle of $X := S^7 \times \mathbb{CP}^q$ in some higher dimensional S^{m+2q+7}. Let $\hat{f}$ denote the self-map of γ of X that covers f, and let $T(\hat{f})$ be the induced self-map of its Thom space $T(\gamma)$. Let $q\colon\ S^N \to T(\gamma)$ be the map that collapses the complement of a tubular neighborhood of X to a point. By definition in [Bro72] (cf. [MTW80]) the refined normal invariant $\mathfrak{q}^t(f) \in [X, F]$ is the Spanier-Whitehead dual of $T(\hat{f})\circ q$, where the set $[X, F]$ of free homotopy classes is identified with the set of based homotopy classes $[X_+, F] = \{X_+, S^0\}$.

The proof of Proposition 7.1 shows that f comes from the $E^2_{1,6}$-term which is in the bottom filtration, which means that it comes from $E^1_{1,6} = \pi_7(\mathrm{Filt}^{(1)}, \mathrm{Filt}^{(0)})$, where $\mathrm{Filt}^{(0)} = \{\mathbf{id}\}$ and $\mathrm{Filt}^{(1)}$ consists of functions in $F_{S^1}(\mathbb{C}^{q+1})$ that restrict to inclusions on the standard subsphere $S^{2q-1} \subset S^{2q+1}$. Changing f within its homotopy class, we can assume that f is identity on $S^7 \times \mathbb{CP}^{q-1} \subset S^7 \times \mathbb{CP}^q$, and as is explained at the end of Section 8, we may also assume that f is identity on a regular neighborhood of $S^7 \vee \mathbb{CP}^q$. In summary, we may suppose that f is identity on the complement of a top-dimensional cell.

Since $p(S^7)\circ f = p(S^7)$, the homotopy self-equivalence f is determined by $p(\mathbb{CP}^q)\circ f$. By the previous paragraph f factors as

$$X \xrightarrow{\text{diag}} X \vee X \xrightarrow{\mathbf{id}\vee\text{pinch}} X \vee S^{2q+7} \xrightarrow{(\mathbf{id},\sigma)} X$$

where $p(S^7)\circ\sigma$ is null-homotopic and $p(\mathbb{CP}^q)\circ\sigma\colon S^{2q+7} \to \mathbb{CP}^q$ represents an element of $\pi_{2q+7}(\mathbb{CP}^q) \cong \pi_{2q+7}(S^{2q+1}) \cong \pi_6^{\mathbf{S}} \cong \mathbb{Z}_2$. The unique nontrivial element of $\pi_6^{\mathbf{S}}$ is ν^2, and $p(\mathbb{CP}^q)\circ\sigma$ is nontrivial, as f corresponds to an order 2 element of $\pi_7(F_{S^1})$. Hence $p(\mathbb{CP}^q)\circ\sigma$ has to factor as $\nu^2\colon S^{2q+7} \to S^{2q+1}$ postcomposed by the bundle projection $S^{2q+1} \to \mathbb{CP}^q$.

The pullback of γ via $(\mathbf{id},\sigma)$ is the vector bundle γ' whose restrictions to the summands of $X \vee S^{2q+7}$ are γ and the trivial ϵ^m bundle over S^{2q+7}. Its Thom space $T(\gamma')$ is glued from $T(\gamma)$ and $T(\epsilon^m) = \Sigma^m(S^{2q+7}_+) \approx S^{m+2q+7} \vee S^m$ along the common copy of S^m, so $T(\gamma') \approx T(\gamma) \vee S^{m+2q+7}$. The map $T(\hat{f})$ factors as

$$T(\gamma) \xrightarrow{\text{diag}} T(\gamma) \vee T(\gamma) \xrightarrow{\mathbf{id}\vee\text{pinch}} T(\gamma) \vee S^{m+2q+7} \xrightarrow{(\mathbf{id},\tilde{\sigma})} T(\gamma)$$

where $\tilde{\sigma}$ is the composition

$$\Sigma^m(S^{2q+7}) \xrightarrow{\Sigma^m(\nu^2)} \Sigma^m(S^{2q+1}) \xrightarrow{\text{inclusion}} \Sigma^m(S^{2q+1}_+) \longrightarrow T(\gamma)$$

where $\Sigma^m(S^{2q+1}_+)$ is the Thom space of $S^{2q+1} \times \mathbb{R}^m$, and the rightmost map is the Thomification of the pullback of γ via the map $S^{2q+1} \to S^7 \times \mathbb{CP}^q$ that is constant on the first factor and the standard projection on the second factor.

The normal invariant of $\mathbf{id}(X)$ is trivial, so $\mathfrak{q}^t(f)$ equals to the S-dual of $\tilde{\sigma}$. The restriction of $\mathfrak{q}^t(f) \in \{X_+, S^0\}$ to $\mathbb{CP}^q{}_+$ is the S-dual of $\tilde{\sigma}$ that is thought of as a map $S^{2q+7} \to T(\check{\gamma})$, where $\check{\gamma}$ is the restriction of γ to the $\mathbb{CP}^q$-factor. Now ν^2 is self-dual, while the S-dual of $\Sigma^m(S^{2q+1}) \hookrightarrow \Sigma^m(S^{2q+1}_+) \to T(\check{\gamma})$ is the Umkehr map $\Sigma(\mathbb{CP}^q{}_+) \to S^{2q+1}_+ \to S^0$ (see [BS74]).

Next we show that the restriction of the Umkehr map to $\Sigma(\mathbb{CP}^1)$ is a generator of $\{\Sigma(\mathbb{CP}^1{}_+), S^0\} \cong \pi_3(F) \cong \pi_3^{\mathbf{S}} \cong \mathbb{Z}_{24}$. Indeed, $\Sigma(\mathbb{CP}^1)$ represents a generator of the $\mathbb{Z}$-factor in

$$\pi_3^{\mathbf{S}}(\Sigma(\mathbb{CP}^\infty_+)) = \pi_3^{\mathbf{S}}(\Sigma(\mathbb{CP}^\infty) \vee S^1)) = \pi_3^{\mathbf{S}}(\Sigma(\mathbb{CP}^\infty)) \oplus \pi_3^{\mathbf{S}}(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

By commutativity of the diagram (6.10) in [BS74] the map $\pi_3^{\mathbf{S}}(\Sigma(\mathbb{CP}^\infty_+)) \to \pi_3^{\mathbf{S}}$ induced by Umkehr coincides with the forgetful map $\pi_3(F_{S^1}) \to \pi_3(F)$. By [BS74, Theorem 11.1] the image of $\pi_3(U) \to \pi_3(F_{S^1})$ induced by inclusion is infinite cyclic. Thus the class of $\Sigma(\mathbb{CP}^1)$ equals to a generator of $\pi_3(U) = \mathbb{Z}$ up to an element of order ≤ 2, hence their images in $\pi_3(F)$ differ by an order 2 element, so it suffices to show that one of them is a generator. That a generator of $\pi_3(U)$ is mapped to a generator of $\pi_3(F)$ is true because $\pi_3(U) \to \pi_3(O)$ is onto as $\pi_3(O/U) = 0$ by Bott periodicity, and the real J-homomorphism $\pi_3(O) \to \pi_3(F)$ is onto.

The group $\pi_3(F) \cong \pi_3^{\mathbf{S}} \cong \mathbb{Z}_{24}$ is also generated by ν. Since ν^3 is nonzero in $\pi_9^S = (\mathbb{Z}_2)^3$ where all nontrivial elements has order 2, so the composition of ν^2 with any generator of $\pi_3^{\mathbf{S}}$ equals to ν^3. Thus $\mathfrak{q}^t(f)$ restricted to $\Sigma\mathbb{CP}^1$ is ν^3.

Finally, $\mathfrak{q}^t(f)$ is mapped to $\mathfrak{q}(f)$ under $[X, F] \to [X, F/O]$, and it remains to show that ν^3 is not in the kernel of $\pi_9(F) \to \pi_9(F/O)$ which equals to the image of the J-homomorphism $\pi_9(O) \to \pi_9(F)$. This image has order two, and as we see below its nonzero element is $\nu^3 + \eta\circ\epsilon \neq \nu^3$, which completes the proof. (To compute the image of the J-homomorphism note that by [Ada66, Theorem 1.2] η^2 induces a nonzero homomorphism $\mathbb{Z} = \pi_7(O) \to \pi_9(O) = \mathbb{Z}_2$. Thus if γ denotes a generator of $\pi_7(O)$, then $\gamma\eta^2$ generates $\pi_9(O)$. The J-homomorphism $\pi_7(O) \to \pi_7(F) \cong \mathbb{Z}_{240}$ is onto, so $\pi_7(F)$ is generated by $J(\gamma)$, and then it follows that the image of $J\colon \pi_9(O) \to \pi_9(F) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is generated by $J(\gamma\eta^2) = J(\gamma)\eta^2$. Tables in [Tod62, pp. 189-190] imply that the image of $\eta^2\colon \pi_7(F) \to \pi_9(F)$ is generated by $\eta^2\sigma = \nu^3 + \eta\circ\epsilon$).

11. Dichotomy principles and skeletal filtrations

We start by proving an important particular case of Theorem 5.1; of course, only one direction is nontrivial.

Proposition 11.1. *For $q \geq 2$ let f be a homotopy self-equivalence of $S^7 \times \mathbb{CP}^q$ that comes from an element of $\pi_7\left(F_{S^1}(\mathbb{C}^{q+1})\right)$. Then f is homotopic to a diffeomorphism if and only if f has trivial normal invariant.*

Proof. Suppose first that $q \geq 3$ so that $\pi_7(F_{S^1}(C^{q+1})) \cong \pi_7(U_{q+1}) \oplus \mathbb{Z}_2$. We write $f = f_1\circ f_2$ where f_1 comes from the $\pi_7(U_{q+1})$-factor and either f_2 comes from the $\mathbb{Z}_2$-factor. Then f_1 is homotopic to a diffeomorphism, because U_{q+1} acts on $\mathbb{CP}^{2m}$ by diffeomorphisms. By Section 10 either f_2 is homotopic to identity, or f_2 has nontrivial normal invariant, and the claim follows.

Suppose now that $q = 2$. By Proposition 7.1 the map f is homotopic to the composition $f_1\circ f_2$ of homotopy self-equivalence f_1, f_2, where each factor has order at most 2, the map f_1 comes from an element in the kernel of $\pi_7(F_{S^1}(\mathbb{C}^3)) \to \pi_1(F_{S^1})$, and f_2 is either homotopic to identity, or

else comes from an element that is mapped to the unique order 2 element of $\pi_1(F_{S^1})$. Corollary 9.2 implies that f_1 is homotopic to a diffeomorphism. Suppose that f_2 is not homotopic to identity. Then as in the proof of Corollary 9.2 $\mathfrak{q}(f_2) \in [S^7 \times \mathbb{CP}^2, F/O]$ is the image of the refined normal invariant $\mathfrak{q}^t(f_2) \in [\Sigma^7(\mathbb{CP}^q) \vee S^7, F]$, which by assumption is the restriction of a stable refined normal invariant $\mathfrak{q}^t \in \{\Sigma^7(\mathbb{CP}^\infty) \vee S^7, F]$ as follows from commutativity of the diagram in the proof of Corollary 9.2. By Section 10 the restriction of $\mathfrak{q}^t$ to $\Sigma^7(\mathbb{CP}^1) \vee S^7$ is nontrivial, and hence the same holds for $\mathfrak{q}^t(f_2)$. $\square$

Lemma 11.2 below leads to a quick proof of Theorem 5.1, and hence Theorem 1.1 in the case $S^7 \times \mathbb{CP}^2$. Indeed, by Propositions 6.1, 6.2, 11.1, and Lemma 11.2, any homotopy self-equivalence of $S^7 \times \mathbb{CP}^2$ either is homotopic to a diffeomorphism, or is the composition of a diffeomorphism with a homotopy self-equivalence that has nontrivial normal invariant, so the result follows from the composition formula for normal invariants. Later in this section we prove Theorem 5.1 in full generality without using Lemma 11.2. Recall that SG_{k+1} is a standard notation for $E_1(S^k)$.

Lemma 11.2. *The group* $[\mathbb{CP}^2, SG_8]$ *is trivial.*

Proof. Recall that the evaluation map defines a fibration $SG_{k+1} \to S^k$, with fiber SF_k, the submonoid of SG_{k+1} consisting of base-preserving maps (see [MM79, Chapter 3A]).

We know that $[\mathbb{CP}^2, S^7]$ is trivial because the dimension of $\mathbb{CP}^2$ is less than the connectivity of S^7, so any map from $\mathbb{CP}^2$ to SG_8 can be homotoped into a fiber of the fibration $SF_7 \to SG_8 \to S^7$. Since SF_7 has the homotopy type of the component of the constant map in the iterated loop space $\Omega^7 S^7$, we get the isomorphism

$$[\mathbb{CP}^2, SF_7] \;\cong\; [S^7\mathbb{CP}^2, S^7]$$

so it suffices to show that the latter vanishes. Now $S^7\mathbb{CP}^2$ is the mapping cone of $S^7\eta_2$, where $\eta_2 : S^3 \to S^2$ is the Hopf map, and therefore we have the following cofiber exact sequence:

$$\pi_{11}(S^7) \longrightarrow [S^7\mathbb{CP}^2, S^7] \longrightarrow \pi_9(S^7) \xrightarrow{(S^7\eta_2)^*} \pi_{10}(S^7)$$

Results on the homotopy groups of spheres [Toda, Chapter XIV] imply that $\pi_{11}(S^7) = 0$, and $(S^7\eta_2)^*$ is injective because $\pi_9(S^7) = \pi_2^{\mathbf{S}}$ is generated by η^2, and $(S^7\eta_2)^*$ stably amounts to composing with η, and $\eta^3 = 4\nu$ is nontrivial in $\pi_3^{\mathbf{S}} = \pi_{10}(S^7)$. Thus $[\mathbb{CP}^2, SF_7]$ must be trivial, as desired. $\square$

The case of $S^7 \times \mathbb{CP}^q$ with $q > 2$ needs more work. We start by proving a Dichotomy Property for normal invariants of homotopy self-equivalences coming from maps $\mathbb{CP}^q \to E_1(S^k) = SG_{k+1}$.

Proposition 11.3. (Dichotomy Property) *Let X be a closed connected smooth n-manifold, let $k \geq 2$ with $n+k \geq 5$, let $u : X \to SG_{k+1}$ be continuous, and let $f\colon S^k \times X \to S^k \times X$ denote the homotopy self-equivalence arising from u. Then either f is homotopic to a diffeomorphism, or else f is not normally cobordant to the identity. In the first case, the diffeomorphism extends to a diffeomorphism of $D^{k+1} \times X$.*

Proof. A key point is that every homotopy self-equivalence of S^k extends to D^{k+1} by the cone construction, which implies that f extends to a homotopy self-equivalence $\hat{f}$ of $D^{k+1} \times X$ and hence yields a homotopy structure on $D^{k+1} \times X$. By Wall's $\pi - \pi$ Theorem [Wal99, Chapter 3], the map $\hat{f}$ is homotopic to a diffeomorphism if and only if its normal invariant is trivial. The restriction map from $[D^{k+1} \times X, F/O] \cong [X, F/O]$ to $[S^k \times X, F/O]$ is split injective, more precisely if $G\colon D^{k+1} \times X \to F/O$ restricts to $g\colon S^k \times X \to F/O$, then $g|_{\{*\}\times X}$ corresponds to G under $[D^{k+1} \times X, F/O] \cong [X, F/O]$. By the geometric definition of normal invariant, $\mathfrak{q}(\hat{f})$ maps to $\mathfrak{q}(f)$ by restriction to the boundary, and therefore, by the previous sentence, $\mathfrak{q}(f)$ maps to $\mathfrak{q}(\hat{f})$ by restriction to $\{*\} \times X$. It follows that $\mathfrak{q}(f)$ is trivial if and only if $\mathfrak{q}(\hat{f})$ is trivial, and if is trivial, Wall's $\pi - \pi$ Theorem implies that $\hat{f}$, and hence f are homotopic to a diffeomorphism. □

One step in the preceding argument is important enough to be stated explicitly: the normal invariant of f lies in the image of $[X, F/O]$ in $[S^k \times X, F/O]$; we shall need a strengthened form of this result.

Corollary 11.4. *Under the assumptions of Proposition 11.3, if A is a subcomplex in some triangulation of X and if the restriction $u|_A$ is trivial in $[A, SG_{n+1}]$, then the restrictions of $\mathfrak{q}(f)$ to A and $S^k \times A$ are also trivial.*

Proof. If B is a closed regular neighborhood of A, then by the Homotopy Extension Property we may replace u with some v in the same homotopy class such that the restriction of v to B is constant (with value 1_X). Let g be the homotopy self-equivalence of $S^k \times X$ that corresponds to g. Then g maps $S^k \times B$ to itself by the identity, and it also maps $X - \mathrm{Int}(B)$ to itself. Standard restriction properties of normal invariants imply that the restriction of the normal invariant of g to $S^k \times B$ is trivial, and this implies the same conclusion for the restriction to $S^k \times A$. □

Now that we have Dichotomy Property for homotopy self-equivalences coming from elements in $\pi_7(E_1(\mathbb{CP}^q))$ and $[\mathbb{CP}^q, E_1(S^7)]$ we only need to see whether their normal invariants can cancel, so that the normal invariant of the composition of these homotopy self-equivalences cannot be trivial unless both summands vanish. This matter is naturally treated in the framework of skeletal filtrations.

Let T be a contravariant functor defined from the homotopy category of pointed finite cell complexes to the category of abelian groups. If X is a pointed finite cell complex, then we say that a a class $u \in T(X)$ *has skeletal filtration* $\geq k$, if the restriction of u to the k-skeleton X_k is trivial, and we say that the *skeletal filtration of* u *equals* k if u has filtration $\geq k$ but does not have filtration $\geq k+1$. The Cellular Approximation Theorem for continuous maps of CW-complexes implies that the skeletal filtration of a class in $T(X)$ does not depend upon the choice of cell decomposition; in fact, it follows that the sets $T^{\langle k\rangle}(X)$ of elements with skeletal filtration $\geq k$ are subgroups and define a filtration of T by subfunctors.

Proposition 11.5. *Suppose that f is a homotopy self-equivalence of $S^7 \times \mathbb{CP}^q$ with $q \geq 2$, which comes from an element of $[\mathbb{CP}^q, SG_8]$. If the normal invariant of f is nontrivial, then its filtration is an even number, which is ≥ 4.*

Proof. In view of the proof of Proposition 11.3, we might as well consider the normal invariant for the homotopy self-equivalence of $D^8 \times \mathbb{CP}^q$ extended via the cone construction, which lies in $[D^8 \times \mathbb{CP}^q, F/O)] \cong [\mathbb{CP}^q, F/O]$. Since $\mathbb{CP}^q$ has cells only in even dimensions, it follows that the filtration of a nontrivial element cannot must be even. Since $[\mathbb{CP}^2, SG_8]$ is trivial, Corollary 11.4 implies that the skeletal filtration of the normal invariant is at least 4. □

Remark 11.6. On the other hand, if f is a homotopy self-equivalence of $S^7 \times \mathbb{CP}^q$ with $q \geq 2$ that comes from an element of $\pi_7(E_1(\mathbb{CP}^q))$, and if $\mathfrak{q}(f)$ is nontrivial, then the skeletal filtration of $\mathfrak{q}(f)$ is odd. Indeed, as in the proof of Proposition 6.2 we may assume that f is identity on $S^7 \vee \mathbb{CP}^q$; hence $\mathfrak{q}(f)$ can be thought of as an element of $[S^7 \wedge \mathbb{CP}^q, F/O]$. Thus if $\mathfrak{q}(f)$ is nontrivial, then the filtration of $\mathfrak{q}(f)$ is odd because $S^7 \wedge \mathbb{CP}^q$ has a cell decomposition (inherited from the product of the standard cell decomposition of $\mathbb{CP}^q$ and $S^k = D^k \cup D^0$) whose positive dimensional cells only appear in odd dimensions from 9 to $2q+7$. In fact, the 9th skeleton of $S^7 \wedge \mathbb{CP}^q$ is $S^7 \wedge \mathbb{CP}^1 = S^9$, and it was shown in Section 10 that the restriction of $\mathfrak{q}(f)$ to $S^7 \wedge \mathbb{CP}^1$ defines a nontrivial element of $\pi_9(F/O)$, so the filtration of $\mathfrak{q}(f)$ is 9.

Proof of Theorem 5.1. By Propositions 6.1, 6.2, any homotopy self-equivalence f of $S^7 \times \mathbb{CP}^2$ either f is homotopic to a diffeomorphism, or $f = f_1 \circ f_2 \circ \phi$ where ϕ is a diffeomorphism and f_1, f_2 are homotopy self-equivalences coming from elements in $\pi_7(E_1(\mathbb{CP}^q))$, $[\mathbb{CP}^q, E_1(S^7)]$, respectively. The composition formula for normal invariants says that

$$\mathfrak{q}(f) = \mathfrak{q}(f_1 \circ f_2) = \mathfrak{q}(f_1) + (f_1)^{*-1}\mathfrak{q}(f_2).$$

By above either f_1 is homotopic to a diffeomorphism, or else the filtration of $\mathfrak{q}(f_1)$ is odd. Similarly, either f_2 is homotopic to a diffeomorphism, or else

its filtration is even; general considerations then imply the same conclusion for $(f_1^*)^{-1}\mathfrak{q}(f_2)$. Therefore, if $\mathfrak{q}(f)$ is trivial, then both $\mathfrak{q}(f_1)$ and $\mathfrak{q}(f_2)$ are trivial, and hence f_1, f_2 are homotopic to diffeomorphisms, so f is homotopic to a diffeomorphism. □

12. Homotopy inertia group of $S^7 \times \mathbb{CP}^2$

Here we obtain an optimal version of Taylor's Theorem 3.2 for $M = S^7 \times \mathbb{CP}^2$.

Theorem 12.1. *The subgroup $I_h(S^7 \times \mathbb{CP}^2) \cap bP_{12}$ has index 4 in bP_{12}. The manifolds $\Sigma(d) \times \mathbb{CP}^2$ fall into 3 diffeomorphism types, and 4 oriented diffeomorphism types.*

It follows from Corollary 5.2 and Remark 5.3 that the first sentence in Theorem 12.1 implies the second one, which proves Theorem 1.2.

That $I_h(S^7 \times \mathbb{CP}^2) \cap bP_{4r}$ contains an index 4 subgroup is a general phenomenon arising from the product formula for the surgery obstruction, and numerical properties of orders of groups bP_{4r} which we denote $|bP_{4r}|$. The following lemma generalizes an argument in [Bro68, (6.5)] given for $m = 1$.

Lemma 12.2. *If m is not divisible by 3, then $\Sigma^{4m+7}(4) \in I_h(S^7 \times \mathbb{CP}^{2m})$, and the manifolds $\Sigma^{4m+7}(d) \times \mathbb{CP}^{2m}$ fall into at most 3 diffeomorphism types, and at most 4 oriented diffeomorphism types.*

Proof. Setting $d = |bP_{k+1}|$ in Fact 3.1, we see that h is homotopic to a diffeomorphism so $\ker(\Delta) \le \mathbb{Z}$ contains the subgroup of index $|bP_{k+1}|$, and similarly $\ker(\Delta)$ contains the subgroup of index $|bP_{4m+k+1}|$.

By [KM63], for $r \ge 2$ the order of bP_{4r} is $a_r 2^{2r-2}(2^{2r-1} - 1)n_r$ where a_r is 2 if r is odd and 1 if r is even, and and n_r is the numerator of $B_r/4r$ where B_r is the corresponding Bernoulli number. Basic results in number theory imply that either $n_r = 1$, or n_r equals to a product of irregular primes.

It is straightforward to check that 7 divides $|bP_{4r}|$ if and only if 3 divides $r-2$. (The point is that 7 does not divide n_r because the smallest irregular prime is 37, and hence we need to see when 7 divides $(2^{2r-1} - 1) \cdot 2$; setting $r = 3s + u$ with $u \in \{0, 1, 2\}$, $s \in \mathbb{Z}$, we get $2^{2r} - 2 = 8^{2s} \cdot 2^{2u} - 2$ which equals to $2^{2u} - 2$ mod 7, so u must be 2.)

Now specialize to the case $k = 7$ for which $|bP_8| = 28 = 4{\cdot}7$, and $bP_{4m+8} = bP_{4r}$ for $r = m+2$, and suppose that 3 does not divide m, so that 7 does not divide $|bP_{4m+8}|$. Since $\ker(\Delta) \le \mathbb{Z}$ contains $28\mathbb{Z}$ and $|bP_{4m+8}|\mathbb{Z}$, it then must contain $4\mathbb{Z}$ as claimed. □

That $I_h(S^7 \times \mathbb{CP}^2) \cap bP_{4r}$ lies in an index 4 subgroup is immediate (by the proof of Taylor's theorem in [Sch87] once the following strengthening of [Sch87, Sublemma 2.3] is obtained.

Lemma 12.3. *If ξ is a stably fiber homotopically trivial vector bundle over the suspension of $S^7 \times \mathbb{CP}^2$, then for each positive integer m the m^{th} Pontryagin class $p_m(\xi)$ is divisible by $2j_{4m}$, where j_{4m} is the order of the image of the J-homomorphism in dimension $4m-1$.*

Proof. For any spaces X, Y there is a natural homotopy equivalence of $\Sigma(X \times Y)$ and $\Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$ (see e.g. [Bro72, proof of III.4.6]). Hence it suffices to establish the result for bundles over S^8, $S\mathbb{CP}^2$, $S^8\mathbb{CP}^2$. Since $S\mathbb{CP}^2$ is obtained by attaching a 5-cell to $S\mathbb{CP}^1 = S^3$ and $\pi_3(BO) = \pi_5(BO) = 0$, we know that $[S\mathbb{CP}^2, BO]$ is trivial.

A key ingredient in what follows is a result of Bott (see [BM58]) that the Pontryagin class p_m of any vector bundle over S^{4m} is divisible by $a_m \cdot (2m-1)!$, where $a_m = 2$ if m is odd and $a_m = 1$ if m is even.

Suppose that ξ is a stably fiber homotopically trivial vector bundle over S^8. Since $\pi_7(F/O) = 0$, the exact homotopy sequence of the fibration implies that $\mathbb{Z} = \pi_8(BO) \to \pi_8(BF) = \mathbb{Z}_{240}$, is onto, so ξ is stably isomorphic to 240η where η represents a generator in $\pi_8(BO)$. By Bott's result $p_8(\eta)$ is divisible by 6. It follows that $p_8(\xi) = 240p_8(\eta)$, hence $p_8(\xi)$ is divisible by $240 \cdot 6$, but $j_8 = 240$, so $p_8(\xi)$ is divisible by $6j_8$, as desired.

Next, suppose that ξ is a stably fiber homotopically trivial vector bundle over $S^8\mathbb{CP}^2$. In the commutative diagram vertical arrows are J-homomorphisms, and rows are cofiber exact sequence associated with the mapping cone sequence $S^2 = \mathbb{CP}^1 \to \mathbb{CP}^2 \to \mathbb{CP}^2/\mathbb{CP}^1 = S^4$.

$$\begin{tikzcd}
\pi_{11}(BO) \arrow[r] \arrow[d] & \pi_{12}(BO) \arrow[r, "\times 2"] \arrow[d, "\text{onto}"] & {[S^8\mathbb{CP}^2, BO]} \arrow[r] \arrow[d] & \pi_{10}(BO) \arrow[r] \arrow[d, "1-1"] & \pi_{11}(BO) \arrow[d] \\
\pi_{11}(BF) \arrow[r] & \pi_{12}(BF) \arrow[r] & {[S^8\mathbb{CP}^2, BF]} \arrow[r] & \pi_{10}(BF) \arrow[r] & \pi_{11}(BF)
\end{tikzcd}$$

One knows that $[\mathbb{CP}^2, BO] = \mathbb{Z}$ [San64, Theorem 3.9], and by Bott Periodicity $\pi_{12}(BO) = \mathbb{Z}$, $\pi_{11}(BO) = 0$, $\pi_{10}(BO) = \mathbb{Z}_2$, and $[S^8\mathbb{CP}^2, BO] = [\mathbb{CP}^2, BO]$. Thus the map

$$[S^8(\mathbb{CP}^2/\mathbb{CP}^1), BO] = \pi_{12}(BO) \to [S^8\mathbb{CP}^2, BO]$$

is multiplication by ± 2.

Since $J\colon \pi_{10}(BO) \to \pi_{10}(BG)$ is one-to-one, and ξ is a stably fiber homotopically trivial, ξ is a pullback of some vector bundle ζ over S^{12}. Since

$J\colon \mathbb{Z} = \pi_{12}(BO) \to \pi_{12}(BG) = \mathbb{Z}_{504}$ is onto, and $\pi_{11}(BF) = \mathbb{Z}_6$, a diagram chase shows that (the class of) ζ in $\pi_{12}(BO) = \mathbb{Z}$ lies in $84\mathbb{Z}$ where $84{\cdot}6 = 504$, so $\zeta = 84\zeta'$ in $\pi_{12}(BO)$. By Bott's result, $p_3(\zeta')$ is divisible by $2 \cdot 5!$, so $p_3(\zeta)$ is divisible by $84 \cdot 2 \cdot 5!$. Recalling that pullback acts as multiplication by 2, we see that $p_3(\xi)$ is divisible by $2 \cdot 84 \cdot 2 \cdot 5! = 80 \cdot 504 = 80 j_{12}$, which completes the proof. □

13. Manifolds tangentially homotopic to $S^7 \times \mathbb{CP}^2$

Theorem 13.1. *If d is an odd integer, and a closed manifold M is tangentially homotopy equivalent to $S^7 \times \mathbb{CP}^2$, then M is diffeomorphic to $S^7 \times \mathbb{CP}^2$, or $\Sigma^7(d) \times \mathbb{CP}^2$, or $\Sigma^7(2d) \times \mathbb{CP}^2$.*

Proof. By Theorem 12.1 it suffices to show that M is diffeomorphic to $\Sigma^7(d) \times \mathbb{CP}^2$ for some d. The key point is to understand the normal invariant $\mathfrak{q}(h)$ of an arbitrary tangential homotopy equivalence $h : M \to S^7 \times \mathbb{CP}^2$. Since h is tangential, $\mathfrak{q}(h)$ is the image of the refined normal invariant $\mathfrak{q}^t(h) \in [S^7 \times \mathbb{CP}^2, F]$.

The exact cofiber sequence for the quotient map $S^7 \times \mathbb{CP}^2 \to S^7 \wedge \mathbb{CP}^2 = S^7\mathbb{CP}^2$

$$[S^7\mathbb{CP}^2, F] \to [S^7 \times \mathbb{CP}^2, F] \to \pi_7(F) \ \oplus \ [\mathbb{CP}^2, F].$$

maps into the similar exact cofiber sequence for $[S^7 \times \mathbb{CP}^2, F/O]$. Both sequences split via precomposing with projections onto S^7 and $\mathbb{CP}^2$-factors, and this forms a commutative diagram in which each of the three components of $[S^7 \times \mathbb{CP}^2, F]$ is mapped into the corresponding component of $[S^7 \times \mathbb{CP}^2, F/O]$.

Since $\mathbb{CP}^2$ is the mapping cone of the Hopf map $\eta_2\colon S^3 \to S^2$, we know that $[\mathbb{CP}^2, F]$ fits into the following exact cofiber sequence exact sequence for the map $\mathbb{CP}^2 \to \mathbb{CP}^2/\mathbb{CP}^1 = S^4$.

$$\pi_4^{\mathbf{S}} \ \to \ [\mathbb{CP}^2, F] \ \to \ \pi_2^{\mathbf{S}} \ \xrightarrow{\eta^*} \ \pi_3^{\mathbf{S}},$$

and hence $[\mathbb{CP}^2, F] = 0$ because $\pi_4^{\mathbf{S}} = 0$, and composition with η induces a monomorphism from $\pi_2^{\mathbf{S}}$ to $\pi_3^{\mathbf{S}}$ as it sends η^2 to $\eta^3 = 4\nu$, which has order 2 (see [Tod62, Chapter XIV]). As $\pi_7(F/O) = 0$, it follows that $\mathfrak{q}(h)$ lies in $[S^7\mathbb{CP}^2, F/O]$, more precisely, $\mathfrak{q}(h)$ is the image of the $[S^7\mathbb{CP}^2, F]$-component of $\mathfrak{q}(h)$, which we denote $\mathfrak{q}^t(h)|_{S^7\mathbb{CP}^2}$.

The rows of the commutative diagram below are exact cofiber sequences for the map $S^7\mathbb{CP}^2 \to S^7(\mathbb{CP}^2/\mathbb{CP}^1) = S^{11}$, and columns are portions of the exact

homotopy sequence of the fibration $p\colon F \to F/O$.

$$\begin{tikzcd}
0=\pi_{11}(F/O) \arrow[r] & {[S^7\mathbb{CP}^2, F/O]} \arrow[r] & \pi_9(F/O) \arrow[r] & \pi_{10}(F/O) \\
\pi_{11}(F) \arrow[r] \arrow[u] & {[S^7\mathbb{CP}^2, F]} \arrow[r] \arrow[u] & \pi_9(F) \arrow[r, "\eta^*"] \arrow[u, "p^*"] & \pi_{10}(F) \arrow[u] \\
 & & \pi_9(O) \arrow[r] \arrow[u, "1\text{-}1"] & \pi_{10}(O)=0 \arrow[u]
\end{tikzcd}$$

Since $\pi_{11}(F/O) = 0$, we identify $[S^7\mathbb{CP}^2, F/O]$ with the kernel of the map $\pi_9(F/O) \to \pi_{10}(F/O)$ so $\mathfrak{q}(h)$ gets identified with $\mathfrak{q}(h)|_{S^7\mathbb{CP}^1} \in \pi_9(F/O)$. Note that p^* maps $\mathfrak{q}^t(h)|_{S^7\mathbb{CP}^1}$ to $\mathfrak{q}(h)|_{S^7\mathbb{CP}^1}$, and by exactness $\mathfrak{q}^t(h)|_{S^7\mathbb{CP}^1} \in \ker(\eta^*)$. Thus the normal invariant of any tangential homotopy equivalence h lies in $p^*(\ker(\eta^*))$.

By [Tod62, Chapter XIV], $\pi_9(F) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with factors generated by ν^3, μ, $\eta\circ\epsilon$, where η^* acts by precomposing with η, which stably and up to sign amounts to postcomposing with η [Tod62, Proposition 3.1]. Using [Tod62, Theorem 14.1] we see that η^* maps ν^3 and $\eta\circ\epsilon$ to zero, while $\eta^*(\mu) = \eta\circ\mu$ is nonzero. The J-homomorphism $\mathbb{Z}_2 = \pi_9(O) \to \pi_9(F)$ is one-to-one, and its image lies in $\ker(\eta^*)$ because $\eta^*\circ J$ factors through $\pi_{10}(O) = 0$. Thus the subgroup $p^*(\ker(\eta^*))$ has order 2.

As we mentioned in Remark 11.6 there exists a tangential homotopy self-equivalence f of $S^7 \times \mathbb{CP}^2$ with such that $\mathfrak{q}(f)|_{S^7\mathbb{CP}^1}$ is nonzero. Since both $\mathfrak{q}(h)$, $\mathfrak{q}(f)$ lie in an order two subgroup, $\mathfrak{q}(h)$ is either trivial or else equal to $\mathfrak{q}(f)$. In the former case $\mathbf{id}(S^7 \times \mathbb{CP}^2)$ and h are in the same bP_{12}-orbit, and the same is true in the latter case for the classes of f and h. Thus in either case M is diffeomorphic to $\Sigma(d) \# (S^7 \times \mathbb{CP}^2)$ for some d, as promised. $\square$

Remark 13.2. By contrast, any closed manifold M that is tangentially homotopy equivalent to $S^3 \times \mathbb{CP}^2$ must be diffeomorphic to $S^3 \times \mathbb{CP}^2$. Indeed, by [MS99, Corollary 4.2] the connected sum of $S^3 \times \mathbb{CP}^2$ with a homotopy sphere is diffeomorphic to $S^3 \times \mathbb{CP}^2$, so it suffices to show that the tangential homotopy equivalence $f\colon M \to S^3 \times \mathbb{CP}^2$ has trivial normal invariant. Now $[S^3\mathbb{CP}^2, F/O] = 0$ because it fits into the exact sequence between the zero groups $\pi_7(F/O)$ and $\pi_5(F/O)$, and moreover, $\pi_3(F/O) = 0$, so the restriction $[S^3 \times \mathbb{CP}^2, F/O] \to [\mathbb{CP}^2, F/O]$ is injective. The claim now follows as $\mathfrak{q}(f)$ comes from $[S^3 \times \mathbb{CP}^2, F]$, and the composition $[S^3 \times \mathbb{CP}^2, F] \to [S^3 \times \mathbb{CP}^2, F/O] \to [\mathbb{CP}^2, F/O]$ factors through $[\mathbb{CP}^2, F] = 0$.

14. Non-diffeomorphic codimension 2 simply-connected souls

In [BKS, Theorem 1.8] the authors showed that if S, S' are closed simply-connected manifolds of dimension ≥ 5 such that complex line bundles over S, S' have diffeomorphic total spaces, then S' is diffeomorphic to the connected sum of S with a homotopy sphere. We prove a partial converse to this statement:

Theorem 14.1. *Let ω be nontrivial complex line bundle over a closed simply-connected n-manifold S with $n \geq 5$, and let S' be the connected sum of S with a homotopy sphere. Let ω' be the pullback of ω via the standard homeomorphism $S' \to S$. Then the disk bundles $D(\omega')$, $D(\omega)$ are diffeomorphic, except possibly when $n \equiv 1 \bmod 4$ and $\pi_1(\partial D(\omega))$ has even order.*

Note that $\pi_1(\partial D(\omega))$ is a finite cyclic group (see Lemma 14.3 below).

Proof of Theorem 14.1. We may assume n is odd, as in even dimensions there is no exotic spheres. By surgery theory the standard homeomorphism $f\colon S' \to S$ has trivial normal invariant in $[S, F/O]$. If $\hat{f}\colon D(f^{\#}\omega) \to D(\omega)$ is the induced map of 2-disk bundles, and if $p\colon D(\omega) \to S$ denotes the disk bundle projection, then the normal invariants of $\hat{f}$ and f are related as $\mathfrak{q}(\hat{f}) = p^*\mathfrak{q}(f)$ as proved e.g. in [BKS, Lemma 5.9]; thus $\mathfrak{q}(\hat{f})$ is trivial. Denote $N := D(\omega)$. Thus the element of the structure set represented by $\hat{f}$ lies in the image of

$$\Delta\colon L^s_{n+3}(\,\pi_1(N), \pi_1(\partial N)\,) \longrightarrow \mathbf{S}^s(N).$$

Lemma 14.3 below implies $\pi_1(\partial N) = \pi_1(S(\omega)) \cong \mathbb{Z}_d$ for some $d \geq 1$, in which case the above relative Wall group $L^s_{n+3}(\,\pi_1(N), \pi_1(\partial N)\,)$ is commonly denoted by $L^s_{n+3}(\mathbb{Z}_d \to 1)$.

If $\mathbb{Z}_d = 1$, then Wall's $\pi - \pi$ theorem implies that $L^s_{n+3}(\mathbb{Z}_d \to 1)$ is trivial, so $\hat{f}$ is homotopic to a diffeomorphism as desired. In general, there is a short exact sequence

$$L^s_{n+3}(\mathbb{Z}_d) \longrightarrow L^s_{n+3}(1) \longrightarrow L^s_{n+3}(\mathbb{Z}_d \to 1) \longrightarrow L^s_{n+2}(\mathbb{Z}_d) \longrightarrow L^s_{n+2}(1)$$

in which the leftmost and rightmost arrows split, via the inclusion $1 \to \mathbb{Z}_d$, thus $L^s_{n+3}(\mathbb{Z}_d \to 1)$ is the kernel of the surjection $L^s_{n+2}(\mathbb{Z}_d) \to L^s_{n+2}(1)$. Results of Wall imply that $L^s_{n+2}(\mathbb{Z}_d) = 0$ if $n \equiv 3 \bmod 4$, and results of Bak and Wall give $L^s_{\mathrm{odd}}(\mathbb{Z}_d) = 0$ if d are odd (see [HT00, p.227]). Thus $\hat{f}$ is homotopic to a diffeomorphism except possibly when $n \equiv 1 \bmod 4$ and d is even, in which case $L^s_{n+3}(\mathbb{Z}_d \to 1) = \mathbb{Z}_2$ as $L^s_{n+2}(\mathbb{Z}_d) = \mathbb{Z}_2$ and $L^s_{n+2}(1) = 0$. □

Proof of Theorem 1.4. Fix homeomorphic, non-diffeomorphic manifolds S, S' that are products $\Sigma^7(d) \times \mathbb{CP}^{2m}$ with $m \geq 1$, or Eschenburg spaces, or Witten

manifolds. Their existence is ensured by Theorem 1.1 and results of Kreck-Stolz mentioned in the introduction.

Then S' is the connected sum of S with a homotopy sphere. For products $\Sigma^7(d)\times\mathbb{CP}^{2m}$ this easily follows as in the proof of Corollary 5.2, and for Eschenburg spaces, or Witten manifolds this is implied by smoothing theory and the fact that their 3rd cohomology with $\mathbb{Z}_2$-coefficients vanish (the point is that if the manifold $\mathring{M}$ obtained by removing an open ball from a closed 7-manifold M with $H^3(M;\mathbb{Z}_2)=0$, then $\mathring{M}$ has a unique PL-structure as $H^3(\mathring{M};\mathbb{Z}_2)=0$, and hence a unique smooth structure as PL/O is 6-connected).

Recall that $\Sigma^7(d)\times\mathbb{CP}^{2m}$ with $m\geq 1$, Eschenburg spaces, and Witten manifolds appear as quotients of $\Sigma^7(d)\times S^{4m+1}$, $SU(3)$, and $S^5\times S^3$, respectively, by free isometric circle actions. Hence they satisfy the assumptions of Lemma 14.2 below. Recall that any element of $H^2(S)\cong\mathbb{Z}$ is the first Chern class of a unique complex line bundle over S. Given a nontrivial line bundle over S, note that by Theorem 14.1 this line bundle and its pullback via the standard homeomorphism $S'\to S$ have diffeomorphic total spaces. By Lemma 14.2 the line bundles have metrics of $\sec\geq 0$ with souls equal to zero sections. $\square$

Proof of Theorems 1.6 and 1.7. Here we deal with the situation where there is a closed n-manifold S with $n\geq 5$ and $n\equiv 3 \bmod 4$ such that for any homotopy sphere $\Sigma^n(d)\in bP_{n+1}$ the connected sum $S\#\Sigma^n(d)$ is the base of a principle circle bundle satisfying assumptions of Lemma 14.2. Fix a nontrivial complex line bundle over S and pull it back to each $S\#\Sigma^n(d)$ via the obvious homeomorphism $S\#\Sigma^n(d)\to S$. As in the above proof of Theorem 1.4 we see that these line bundles have diffeomorphic total spaces that admit complete metrics of $\sec\geq 0$ such that the zero sections are souls. Denote the common total space of all these bundles by N.

Suppose that S' is a soul of an arbitrary metric of $\sec\geq 0$ on N. There is a canonical homotopy equivalence $f_d\colon S\#\Sigma^n(d)\to S'$ given by the inclusion $S\#\Sigma^n(d)\to N$ followed by the normal bundle projection $N\to S'$. It was shown in [BKS, Corollary 5.2, Proposition 5.4] that f_d has trivial normal invariant in $[S',F/O]$, so for some d_0 there is a diffeomorphism $\phi\colon S_0:=S\#\Sigma(d_0)\to S'$.

By [BKS, Corollary 5.2] the Euler classes of the normal bundles of S_0, S' are preserved by f_{d_0}, and since $H^2(S_0)\cong H^2(S')=\mathbb{Z}$, their Euler classes are also preserved by ϕ up to sign. So after changing orientation if needed, we may conclude that ϕ preserves the Euler classes of the normal bundles of S_0, S' in N, and hence the normal bundles themselves, so the pairs (N,S_0) and (N,S') are diffeomorphic. $\square$

Lemma 14.2. *Let $P\to B$ be a principal circle bundle whose total space P is 2-connected and carries an S^1-invariant metric of* $\sec\geq 0$. *Then* $H^2(B)\cong\mathbb{Z}$

and the total space of any complex line bundle over B carries a complete metric of $\sec \geq 0$ *such that the zero section is a soul.*

Proof. By the homotopy sequence of the fibration $P \to B$ we see that B is simply-connected and $\pi_2(B) = \mathbb{Z}$. So by Hurewicz and universal coefficients theorems we get $H^2(B) = \mathbb{Z}$. Note that any complex line bundle ω over B can be written as $P \times_\rho \mathbb{C}$ for some representation $\rho\colon S^1 \to U(1)$ (because vanishing of $H^2(P)$ implies triviality of the pullback of ω via the projection $P \to B$, and ρ comes from the S^1-action on the $\mathbb{C}$-factor of $P \times \mathbb{C}$). The product metric on $P \times \mathbb{C}$ has $\sec \geq 0$ and it descends to a complete metric on $P \times_\rho \mathbb{C}$ of $\sec \geq 0$ with soul $P \times_\rho \{0\}$ which can be identified with B. □

Lemma 14.3. *Let B be a simply-connected closed manifold and let $P \to B$ be the projection of a nontrivial circle bundle. Then $\pi_1(P) \cong \mathbb{Z}_d$, where the Euler class of the circle bundle is d^{th} multiple of a primitive element in the free abelian group $H^2(B)$.*

Proof. Since $H_1(B)$ is trivial, a universal coefficients theorem gives $H^2(B) = \mathrm{Hom}(H_2(B), \mathbb{Z})$, so $H^2(B)$ is free abelian. A portion of Gysin sequence reads

$$0 = H^1(B) \to H^1(P) \to H^0(B) \to H^2(B) \to H^2(P) \to H^1(B) = 0$$

where the middle map is multiplication by the Euler class. Since the Euler class is nontrivial and $H^2(B)$ has no torsion, we see that Euler class has infinite order, so $\mathbb{Z} = H^0(B) \to H^2(B)$ is injective. Thus $H^1(P) = 0$. By another universal coefficients theorem $H_1(P)$ is mapped onto $\mathrm{Hom}(H^1(P), \mathbb{Z}) = 0$ with kernel $\mathrm{Ext}(H^2(P), \mathbb{Z})$, which is isomorphic to the torsion subgroup of $H^2(P)$. Finally homotopy sequence of the circle bundle $P \to B$ and triviality of $\pi_1(B)$ implies that $\pi_1(P)$ is cyclic, and in particular, abelian, so $\pi_1(P) = H_1(P)$. If the Euler class of the circle bundle is dth multiple of a primitive element in $H^2(B)$, then the above Gysin sequence implies that the torsion subgroup of $H^2(P)$ is isomorphic to $\mathbb{Z}_d$, so $\pi_1(P) \cong \mathbb{Z}_d$. □

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Igor Belegradek, School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160

E-mail address: `ib@math.gatech.edu`

Sławomir Kwasik, Mathematics Department, Tulane University, 6823 St. Charles Ave, New Orleans, LA 70118

E-mail address: `kwasik@tulane.edu`

Reinhard Schultz, Department of Mathematics, University of California Riverside, 900 Big Springs Drive, CA 92521

E-mail address: `schultz@math.ucr.edu`